MANONMANIAM SUNDARANAR UNIVERSITY

TIRUNELVELI-627 012

DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

III B.Sc. MATHEMATICS

SEMESTER VI

CORE - XV : MECHANICS

Sub. Code: JMMA63

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MECHANICS (JMMA63)

UNIT	DETAILS
I	Force : Newton's Laws of motion – Resultant of two forces on a particle – Equilibrium of a particle : Equilibrium of a particle – Limiting equilibrium of a particle on an inclined plane.
II	Forces on a Rigid Body : Moment of a Force – General motion of a body – Equivalent system of forces – Parallel forces – Forces acting along a triangle – A specific reduction of Forces : Reduction of coplanar forces into a force and couple – Problems involving frictional forces.
III	Work, Energy and power : Work – Conservative field of force – Power – Rectilinear Motion under varying Force : Simple Harmonic Motion – along a horizontal line – along a vertical line.
IV	Projectiles : Forces on a projectile – Projectile projected on a inclined plane.
V	Central Orbits : General orbits – Central orbit – Conic as a centered orbit.

Text Book
P. Duraipandian, Laxmi Duraipandian and Muthamizh Jayapragasm, Mechanics, S. Chand & Company Ltd., 2007.

UNIT I

FORCES

1.1 NEWTON'S LAWS OF MOTION

Newton enunciated three laws of motion which are used as axioms for developing the subject.

These laws are:

N.1. A particle remains at rest or in a state of uniform motion in a straight line unless acted on by an impressed force.

N.2. The rate of change of momentum of a particle is proportional to the impressed force and it is in the direction of the force.

N.3. To every action there is an equal and opposite reaction.

1.1.1 Forces

The first law, N.1., means that a particle cannot change its uniform motion in a straight line or its state of rest, on its own accord. That is , to effect a change from the uniform motion of a particle in a straight line or from its state of rest, an external agent is necessary and this agent is called force.

Linear Momentum : If m is the mass of the particle and v its velocity, then mv is called the linear momentum or simply the momentum of the particle. If r is the position vector of the particle, then its linear momentum is $m\dot{r}$.

N.1.establishes the existence of the force. We shall now show that N.2enables us to measure it.

Measuring a Force: Let the position vector of a particle of mass m be r. Then by N.2., we get$\frac{d(m\dot{r})}{dt}$=k$\bar{F}$ or $m\ddot{r}$=k$\bar{F}$

where k is a constant and $\bar{F}$ is the impressed force. Now denoting the acceleration $\ddot{r}$by $\bar{a}$, we may rewrite this equation as

$$m\bar{a} = k\bar{F}$$

Further, if $\bar{a} = a e$ and $\bar{F}$=F$\hat{e}$,where $\hat{e}$is the unit vector in the direction of the acceleration, then the equation takes the form

$$ma\hat{e} = kF\hat{e}$$

Let us now define the unit force as that force which produces a unit acceleration on a particle of unit mass. By this choice, we get that, when F=1 and m=1, a=1. Consequently k becomes 1 and N.2 results in the equation.

$$m\bar{a}=\bar{F} \quad \text{or} \quad m\ddot{r}=\bar{F}$$

which is called the equation of motion of the particle. This equation enables us to measure force. Its scalar form is ma=F.

Units of Force: In the M.K.S., C.G.S., and F.P.S. systems, the units of force are respectively a newton, a dyne and a poundal.

Newton: A newton is that force which, acting on a particle of mass 1 kg, produces an acceleration of 1 m/sec^2.

Dyne: A dyne is that force which, acting on a particle of mass 1 gram, produces on it an acceleration of 1 cm./sec^2

Poundal: A poundal is that force which, acting on a particle of mass 1 lb., produces on it an acceleration of 1 ft./sec ^{2}

Remark. 1.newton=10^5 dynes

1.1.2 Types of forces.

The forces with which we will be concerned in this book are earth's gravitation, tension, reaction and resistance and we will be omitting the forces like electromagnetic and magnetostatic forces.

1**) Earth's gravitation**. It is Newton who found that two particles attract each other with a force of attraction whose magnitude is

$$\frac{\gamma m_1 m_2}{r^2}$$

where m_1 and m_2 are the masses of the particles r, the distance between them and γ, a universal constant.

Weight. The force of attraction of the earth on a body is called the weight of the body.

This force is towards the centre of the earth, that is, vertically downwards. For different places on the earth's surface or in the same place but at greatly differing altitudes, g varies slightly and the force of attraction on a given particle correspondingly varies. The acceleration due to this force, close to the surface of the earth, has been observed through experiments conducted in vacuum, to be about 9.8metre/sec^2, 980 cm/sec^2, 32 feet/sec^{2}respectively in the M.K.S., C.G.S. and F.P.S. systems. Now, denoting this acceleration by g, we get the weight of a body of mass m to be mg. So the weights of bodies of masses m kilograms, m grams and m pounds are respectively.

$m \times 9.8$ newtons, $m \times 980$ dynes, $m \times 32$ poundals.

Definition. The weights of a kilogram, a gram, a pound, a tonne (metric ton) and a ton are respectively called a kilogram weight, a gram weight, a pound weight, a tonne weight and a ton weight.

Note. It is erroneous to say that the weight of a man is 64 kg. Technically his mass is 64 kg, and his weight is 64 kilogram weight.

(ii) Tension. Tension is a force which comes into play when an elastic body is deformed by application of forces.

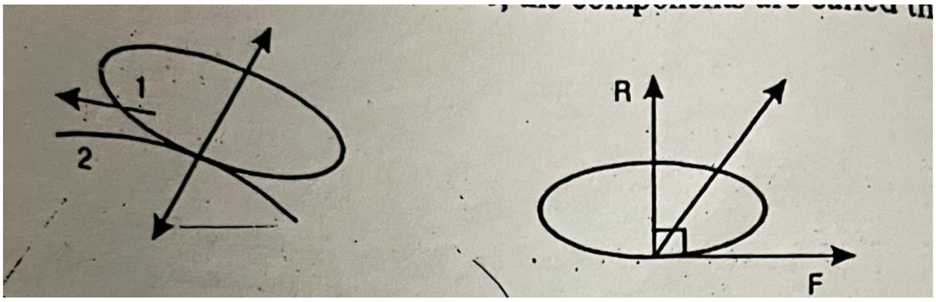
Hooke's law. It has been found by experiments that tension of an elastic string or a spiral spring varies as the ratio of the extension of the string beyond its natural length, to its natural length. This fact was discovered by Hooke and hence this law is known as *Hooke's law*. The constant of proportionality is called the coefficient of elasticity of the string or the modulus of elasticity of the string. This coefficient is usually denoted by λ so that

$$\text{tension} = \lambda \frac{\text{extension}}{natural\ length}$$

When a string is taut, the tension on it is the same everywhere. When it lies over a smooth object, then also the tension on it is the same everywhere.

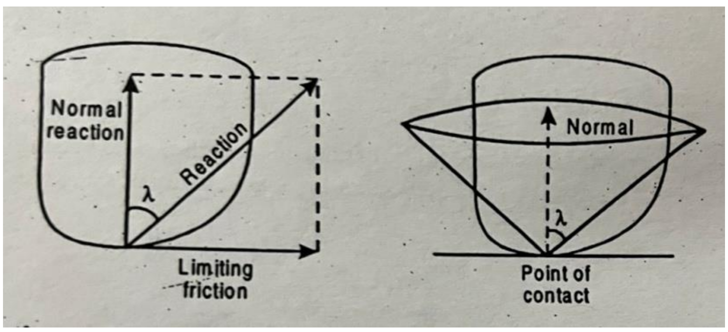
(iii) Reaction. It is observed from experiments that whenever two bodies have contact with each other, each is subject to the action of a force, the forces being equal in magnitude but opposite in direction. This fact was enunciated by Newton as N.3. These forces are called the reactions of the bodies.

Normal Reaction and Friction: We shall now consider the reaction acting on one of these bodies. When this force can be resolved into two components: one along the direction normal to the surface of the body and the other along the direction opposite to the tangential direction in which the body has a tendency to move. These components are called the normal reaction and the friction respectively.It is important to note the difference between reaction and the normal reaction. When the bodies are smooth ,which, of course, is an ideal case, the friction itself vanishes and the reaction is the normal reaction itself.



Limiting Friction: It has been observed through experiments that there is a limit to the amount of friction that can be called into play. This maximum limit is called the limiting friction. When one body is just on the point of sliding on another body, the equilibrium is said to be limiting equilibrium and the friction then exerted is the limiting friction. It has also been observed that the limiting friction bears a constant ratio to the normal reaction. This constant is called the coefficient of friction.If the limiting friction is F and the normal reaction is R and if the coefficient of friction is μ, then:

$$\frac{F}{R} = \mu \quad \text{or} \quad F = \mu R.$$



Angle of Friction and Cone of Friction: When the friction is the limiting friction, the angle between the reaction and the normal to the surface is called the angle of friction. The right circular cone with its vertex at the point of contact, with its axis along the normal to the surface and with its semi-vertical angle equal to the angle of friction, is called the cone of friction.If the angle of friction is λ, then:

$$\tan\lambda = \frac{\text{limiting friction}}{\text{normal reaction}} = \mu \quad \text{or} \quad \lambda = \tan^{-1}\mu$$

Laws of Friction: It is observed from experiments that the nature of friction is governed by the following laws of friction:

Law 1: The friction acts opposite to the direction in which the body moves or has a tendency to move relative to the other body in contact.

Law 2: When the bodies are at rest, the friction called into play is just sufficient to prevent the motion of any of the bodies.

Law 3: There is a limit to the amount of friction that can be called into play. This limit bears a constant ratio to the normal reaction.This constant depends upon the nature and material of the surfaces in contact and not on the measure of the areas in contact.

Law 4: When the body slides on another, the friction called into play is slightly less than the limiting friction.

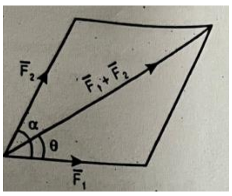
Law 5: When the body slides on another, the friction called into play is independent of its relative speed.

Of the above five laws, the first three are called the laws of static friction and the other two are called the laws of dynamic friction.

1.2 RESULTANT OF TWO FORCES ON A PARTICLE

If a particle is acted upon by two forces F_1 and F_2, then their vector sum$\bar{F}_1 + \bar{F}_2$is called the resultant force on the particle.

Bookwork1.1: To find the Magnitude and direction of the Resultant of$\bar{F}_1$and$\bar{F}_2$.



Now the resultant is$\bar{F}_1+\bar{F}_2$.Let the angle between $\bar{F}_1$ and $\bar{F}_2$ be α. Then the magnitude of

$\bar{F}_1+\bar{F}_2$ is: $|\bar{F}_1+\bar{F}_2| = \sqrt{(\bar{F}_1+\bar{F}_2).(\bar{F}_1+\bar{F}_2)}$

$=\sqrt{(\bar{F}_1.\bar{F}_1+\bar{F}_2.\,\bar{F}_2+2\bar{F}_1.\bar{F}_2)}$

$=\sqrt{\bar{F}_1{}^2+\bar{F}_2{}^2+2\bar{F}_1.\bar{F}_2\cos\alpha}$

Where $|\bar{F}_1|=F_1, |\bar{F}_2|=F_2$

Let θ be the angle between $\bar{F}_1$ and the resultant $\bar{F}_1+\bar{F}_2$. Then:

$$\tan\theta=\frac{|\overline{F1\times(F1}+\bar{F}2)|}{\bar{F}1.\overline{(F}1+\bar{F}2)}$$

$=|0+\bar{F}_1\times\bar{F}_2|/\bar{F}_1{}^2+\bar{F}_1.\bar{F}_2\cos\alpha$

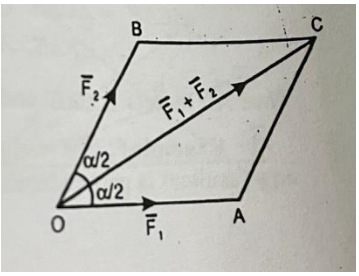
$$=\frac{|F^1F^2\sin\alpha\hat{n}|}{F_1(F_1+F_2\cos\alpha)}$$

$$=\frac{(F_1F_2\sin\alpha)}{F_1(F_1+F_2\cos\alpha)}$$

$$=\frac{(F_2\sin\alpha)}{(F_1+F_2\cos\alpha)}$$

$$\theta=\tan^{-1}\frac{(F_2\sin\alpha)}{(F_1+F_2\cos\alpha)}$$

Corollary 1. If $\bar{F}_1$ and $\bar{F}_2$ are of equal magnitude, say F, then



$|\bar{F}_1+\bar{F}_2|=\sqrt{\bar{F}^2+\bar{F}^2+2\bar{F}^2\cos\alpha}$

$=F\sqrt{2(1+\cos\alpha)}$

$=F\sqrt{4\cos^2\frac{\alpha}{2}}$

$=2F\cos\frac{\alpha}{2}$

Corollary 2. If $\bar{F}_1$ and $\bar{F}_2$ are perpendicular to each other, then choosing $\bar{\imath}$ and $\bar{\jmath}$ in their directions.

$\bar{F}_1=F_1\bar{\imath}$, $\bar{F}_2=F_2\bar{\jmath}$.

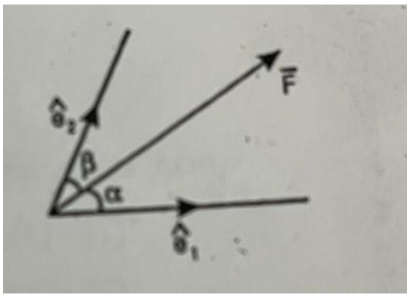
$|\bar{F}_1+\bar{F}_2|=|F_1\bar{\imath}+F_2\bar{\jmath}|$

$$=\sqrt{\bar{F}_1^2+\bar{F}_2^2}$$

$\tan\theta=\frac{F_1}{F^2}$ or $\theta=\tan^{-1}\frac{F1}{F_2}$

1.3 Resolution of a Force into its components. Given the forces $\bar{F}_1$ and $\bar{F}_2$, we have the resultant as $\bar{F}_1+\bar{F}_2$. Conversely, if $\bar{F}_1+\bar{F}_2$ is given, the quantities $\bar{F}_1$ and $\bar{F}_2$ are said to be components of $|\bar{F}_1+\bar{F}_2|$. Since infinite number of parallelograms can be formed with a given diagonal, a given force can be resolved intotwo components in infinite number of different ways.

Bookwork 1.2. To resolve a force $\bar{F}$ into components in two given directions.



Let $\hat{e}_1$, $\hat{e}_2$ be the unit vectors in the given directions. Let them make angles α, β with $\bar{F}$ may be expressed as a linear combination of $\hat{e}_1$, $\hat{e}_2$ as

$$\bar{F}= a\, \hat{e}_1 + b\, \hat{e}_2 \quad ...(1)$$

Multiplying this vectorially by $\hat{e}_1$,

$$\hat{e}_1 \times \bar{F}= a\, \hat{e}_1 \times \hat{e}_1 + b\, \hat{e}_1 \times \hat{e}_2$$

$$\text{i.e.,} F \sin \alpha\, \hat{n} = b \sin(\alpha + \beta)\, \hat{n}$$

where $\hat{n}$ is the unit vector perpendicular to both $\hat{e}_1$ and $\hat{e}_2$ such that $\hat{e}_1$, $\hat{e}_2$, $\hat{n}$ form a right-handed triad.

$$b = \frac{F\sin\alpha}{\sin(\alpha+\beta)}$$

Multiplying (1) vectorially by $\hat{e}_2$,

$$\hat{e}_2 \times \bar{F} = a\, \hat{e}_2 \times \hat{e}_1 + b\, \hat{e}_2 \times \hat{e}_2$$

$$\text{i.e.,} \bar{F} \sin \beta\, (-\hat{n}) = a \sin(\alpha + \beta)\, (-\hat{n})$$

$$a = \frac{F\sin\beta}{\sin(\alpha+\beta)}$$

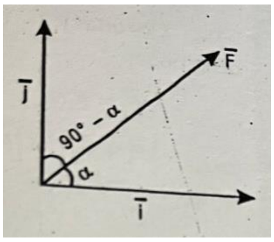
$$\therefore \bar{F} = \frac{F\sin\beta}{\sin(\alpha+\beta)}\hat{e}_1 + \frac{F\sin\alpha}{\sin(\alpha+\beta)}\hat{e}_2$$

1.4 Component of a force in a given direction. Given a force $\bar{F}$ and a direction specified by the unit vector ê, the scalar quantity

$$\bar{F} \cdot \hat{e}$$

is the component of $\bar{F}$ in the direction of ê.

Bookwork 1.3. To express the force F acting in a plane in terms of its components in two perpendicular directions in the plane.



Let $\bar{\imath}, \bar{\jmath}$ be the unit vectors along the perpendicular directions.

Then, from Vector Algebra, we have

$$\bar{F} = (\bar{F} \cdot \bar{\imath})\,\bar{\imath} + (\bar{F} \cdot \bar{\jmath})\,\bar{\jmath}.$$

If $\bar{F}$ makes angles α, 90° − α with $\bar{\imath}, \bar{\jmath}$ then

$$\bar{F} \cdot \bar{\imath} = F.1.\cos\alpha,$$

$$\bar{F} \cdot \bar{\jmath} = F.1.\cos(90° - \alpha) = F\sin\alpha.$$

$$\therefore \bar{F} = F\cos\alpha\,\bar{\imath} + F\sin\alpha\,\bar{\jmath}.$$

Note. F cos α, F sin α are the components of $\bar{F}$ in the perpendicular directions which make angles α, 90° − α with $\bar{F}$.

EXAMPLES

Example 1. The magnitude of the resultant of two given forces P, Q is R. If Q is doubled, then R is doubled. If Q is reversed, then also R is doubled. Show that

$$P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}.$$

Let α be the angle between P, Q. Then, considering the magnitudes of the resultants in the three cases,

$$P^2 + Q^2 + 2PQ \cos \alpha = R^2 \quad ...(1)$$

$$P^2 + 4Q^2 + 4PQ \cos \alpha = 4R^2 ...(2)$$

$$P^2 + Q^2 - 2PQ \cos \alpha = 4R^2 ...(3)$$

where, in the third case, the angle between the forces is $180° - \alpha$.

$$(1) + (3): 2P^2 + 2Q^2 = 5R^2$$

$$(2) + 2(3): P^2 + 2Q^2 = 4R^2$$

$$\therefore P^2 = R^2, Q^2 = 3/2\ R^2$$

$$\therefore P^2 : Q^2 : R^2 = R^2 : 3/2\ R^2 : R^2 = 2 : 3 : 2$$

Example 2. The magnitude of the resultant of the forces $\bar{F}_1$ and $\bar{F}_2$ acting on a particle is equal to the magnitude of $\bar{F}_1$. When the first force is doubled, show that the new resultant is perpendicular to $\bar{F}_2$.

Since the magnitude of $\bar{F}_1+\bar{F}_2$ is equal to the magnitude of $\bar{F}_1$.

$$|\bar{F}_1+\bar{F}_2| = \bar{F}_1 \qquad \text{or} \qquad |\bar{F}_1+\bar{F}_2|2 = |\bar{F}_1|^2$$

$$(\bar{F}_1+\bar{F}_2).(\bar{F}_1+\bar{F}_2) = \bar{F}_1.\bar{F}_1$$

$$\text{or } \bar{F}_1.\bar{F}_1+2\bar{F}_1.\bar{F}_2 + \bar{F}_2.\bar{F}_2$$

$$\text{or } (2\bar{F}_1+\bar{F}_2).\bar{F}_2=0$$

So the resultant of $2\bar{F}_1$ and $\bar{F}_2$ is perpendicular to $\bar{F}_2$.

Example 3. Two forces of equal magnitudes act on a particle and they include an angle θ. If one of them is halved, the angle between the other and the original resultant is bisected by the new resultant. Show that θ = 120°.

Let OA, OB be the given forces. Complete the parallelogram OACB. Since OA = OB, OC bisects ∠AOB. Let B′ be the midpoint of OB. Complete the parallelogram OAC′B′. Now C′ is the midpoint of AC. Since OC′ bisects ∠AOC,

$$\frac{OA}{OC}=\frac{AC'}{CC'}=1$$

So OA = OC. But OA = OB. Hence ΔOCA is an equilateral triangle and

$$\angle AOC = 60° \therefore \angle AOB = 120°$$

Example 4: Two forces of magnitudes P and Q act at a point. Their resultant is inclined to the first at an angle α and has a magnitude R. If the magnitude of the first force is increased by R, show that the new resultant will make an angle $\alpha/2$ with the first force.

Let the given forces be $P\hat{e}_1$, $Q\hat{e}_2$, where $\hat{e}_1$, $\hat{e}_2$ are the unit vectors in the directions of the forces.Then their resultant is

$$P\,\hat{e}_1 + Q\,\hat{e}_2 = R\,\hat{e}_3, \text{ say}$$

since the magnitude is R ($\hat{e}_3$ is the respective unit vector). It is given that the angle between $\hat{e}_1$ and $\hat{e}_3$ is α.

Next, P is increased by R and so the new resultant is

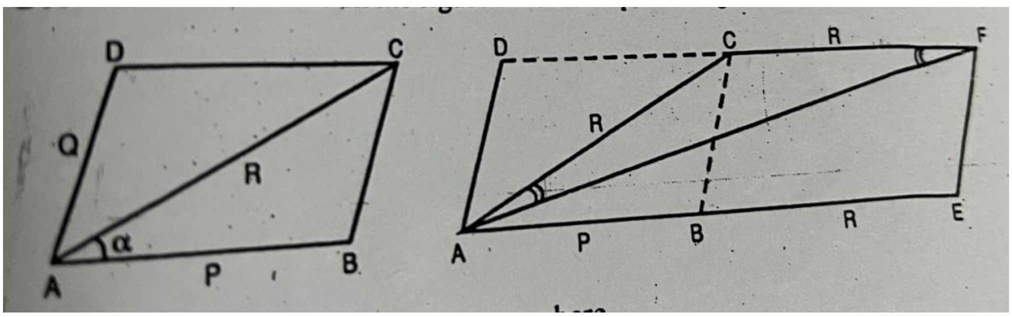
$$(P+R)\,\hat{e}_1 + Q\,\hat{e}_2 = R\hat{e}_1 + (P\,\hat{e}_1 + Q\,\hat{e}_2)$$

$$= R\hat{e}_1 + R\hat{e}_3 = R\,(\hat{e}_1 + \hat{e}_3)$$

But the direction of $\hat{e}_1 + \hat{e}_3$ is the direction which bisects the angle between $\hat{e}_1$ and $\hat{e}_3$, which is α.So the angle between the new resultant and the first force $P\hat{e}_1$ is $\alpha/2$.

Geometrical Method: In the figure ABCD is a parallelogram, AB = P, AD = Q, AC = R and $\angle BAC = \alpha$.Also AEFD is a parallelogram,

Where R=BE=CF=AC



So ΔAFC is isosceles and

$$\angle CAF = \angle CFA \text{(Base angles are equal)}$$

$$= \angle FAE \text{ Alternate angle}$$

Thus AF bisects ∠CAE and so ∠FAE= $\alpha/2$.

Example 5: The resultant of two forces P, Q is of magnitude P. Show that, if P is doubled, the new resultant is perpendicular to the force Q and its magnitude is $\sqrt{4P^2 - Q^2}$.

Denote the first two forces by $\overline{P}$, $\overline{Q}$. Then their resultant and the new resultant are $\overline{P} + \overline{Q}$, $2\overline{P} + \overline{Q}$.

$$|\overline{P} + \overline{Q}|^2 = |\overline{P}|^2 \text{ or } (\overline{P} + \overline{Q}).(\overline{P} + \overline{Q}) = \overline{P}.\ \overline{P}$$

$$\overline{P}.\overline{P} + 2\overline{P}.\ \overline{Q} + \overline{Q}.\ \overline{Q} = \overline{P}.\ \overline{P} \text{ or } 2\overline{P}.\ \overline{Q} + \overline{Q}.\ \overline{Q} = 0 \quad \ldots\ldots..\{1\}$$

So the new resultant is perpendicular to $\overline{Q}$. Also,

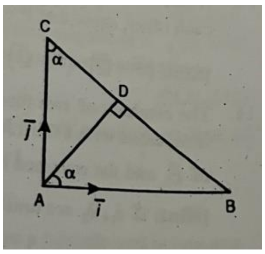
$$|2\overline{P} + \overline{Q}|^2 = (2\overline{P} + \overline{Q}).(2\overline{P} + \overline{Q})$$

$$= (2\overline{P} + \overline{Q}).\ (2\overline{P}) + (2\overline{P} + \overline{Q}).\ \overline{Q}$$

$$=(4P^2+2\overline{P}.\overline{Q})+0=4P^2 - Q^2 \text{ by } (1)$$

$$|2\overline{P} + \overline{Q}| = \sqrt{4P^2 - Q^2}.$$

Example 6: ABC is a triangle, right-angled at A and AD is the perpendicular from A to BC. Show that the resultant of the forces acting along AB, AC with magnitudes 1/AB ,1/AC acts along AD and its magnitude is 1/AD.



Let $\bar{\imath}, \bar{\jmath}$ be the unit vectors along AB, AC. Let ∠BAD=α

Now the unit vector along AD is

$$\hat{AD} = \cos\alpha\ \hat{\imath} + \sin\alpha\ \hat{\jmath}$$

$$= (AD / AB)\ \hat{\imath} + (AD / AC)\ \hat{\jmath} \qquad ...(1)$$

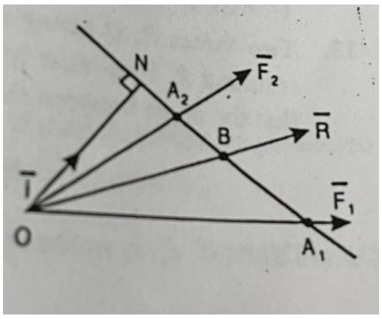
Now the forces are $(1 / AB)\ \hat{\imath}$, $(1 / AC)\ \hat{\jmath}$ and their resultant is

$$(1 / AB)\ \hat{\imath} + (1 / AC)\ \hat{\jmath} = (1 / AD)\ [\ (AD / AB)\ \hat{\imath} + (AD / AC)\ \hat{\jmath}\ ]$$

$= (1 / AD)\, \hat{AD}$ by (1)

which is along AD with a magnitude 1 / AD.

Example 7: The resultant of the forces F_1, F_2 acting at O is R.If any transversal meets the lines of action of F_1, F_2, Rat A_1, A_2, B,prove that $\frac{F_1}{OA_1} + \frac{F_2}{OA_2} = \frac{R}{OB}$.



Draw ON, the perpendicular to the transversal.Let ĵ be the unit vector along ON.Let the forces and the resultant, in vectors, be $\bar{F}_1$, $\bar{F}_2$, $\bar{R}$.

Then,$\bar{F}_1 + \bar{F}_2 = \bar{R}$ ...(1)

The angles between $\bar{\imath}$ and the forces CCC, R are

$\angle A_1ON$, $\angle A_2ON$, $\angle BON$.

Multiplying (1) by $\bar{\imath}$scalarly, we get

$\bar{F}_1.\bar{\imath} + \bar{F}_2.\bar{\imath} = \bar{R}.\bar{\imath}$

$\therefore F_1 \cos A_1ON + F_2 \cos A_2ON = R \cos BON$

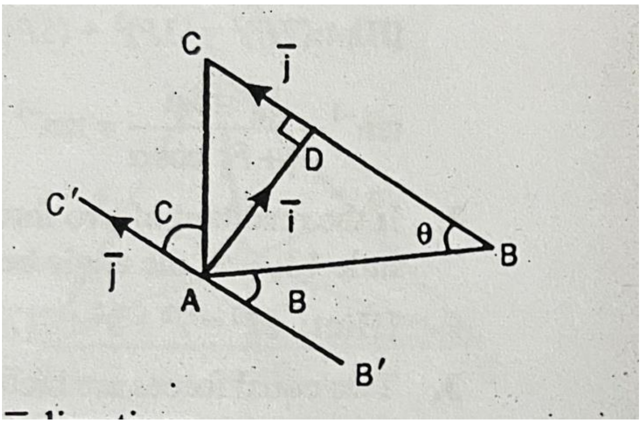
i.e.,$F_1 \frac{ON}{OA_1} + F_2 \cos \frac{ON}{OA_2} = R \cos \frac{ON}{OB}$ or $F_1 \frac{1}{OA_1} + F_2 \frac{1}{OA_2} = R \frac{1}{OB}$

Remark. This result can be extended to n forces $\bar{F}_1, \bar{F}_2, \ldots\ldots. \bar{F}_n$acting at O. In this case we have

$\lambda_1 \frac{F1}{OA1} + \lambda_2 \frac{F2}{OA2} + \ldots\ldots\ldots. + \lambda_n \frac{Fn}{OAn} = \frac{R}{OB}$

where λ_r=1 or -1 according as $\cos A_iON$ is +ve or −ve.

Example 8: Show that the resultant of two forces sec B, sec C acting along the sides AB, AC of a triangle ABC is a forcetan B + tan C acting along AD, where D is the foot of the perpendicular from A on BC.



Draw B′AC′ parallel to BC. Then

∠B′AB = B, ∠C′AC = C.

Let the unit vectors along AD, AC′ be $\bar{\imath}$, $\bar{\jmath}$.Now the force sec B is along AB. Resolving it into components in the $-\bar{\jmath} and \bar{\imath}$, directions, it is

sec B cos B (-$\bar{\jmath}$)+ sec B sin B $\bar{\imath}$or-$\bar{\jmath}$+ tan B$\bar{\imath}$(1)

Resolving sec C, which is along AC, into components in the $\bar{\imath}$ and $\bar{\jmath}$directions

sec C cos C$\bar{\jmath}$ + sec C sin C$\bar{\imath}$or$\bar{\jmath}$ + tan C$\bar{\imath}$(2)

Adding (1) and (2), we get the result as stated.

Example 9: Two forces of magnitudes F_1 and F_2 act at a point. They are inclined at an angle α.If the forces are interchanged, show that their resultant is turned through the angle

$$2\tan^{-1}\left(\frac{F^1-F_2}{F^1+F_2.}\tan\frac{\alpha}{2}\right)$$

Let the forces be $F_1\,\hat{e}_1$ and $F_2\,\hat{e}_2$. Then $\hat{e}_1.\,\hat{e}_2=\cos\alpha$.After the interchange, the forces will be represented by the vectors$F_2\,\hat{e}_1$ and $F_1\,\hat{e}_2$.So the resultants before and after the interchange are respectively $F_1\,\hat{e}_1+F_2\,\hat{e}_2$and $F_2\,\hat{e}_1+F_1\hat{e}_2$.If θ is the angle between them, then

$$\cos\theta=\frac{(F_1\,\hat{e}_1+F_2\,\hat{e}_2)}{|F_2\,\hat{e}_1+F_1\,\hat{e}_2|}\cdot\frac{(F_2\,\hat{e}_1+F_1\,\hat{e}_2)}{|F_2\,\hat{e}_1+F_1\,\hat{e}_2|}$$

$$=\frac{(2F_1F_2+(F_1^2+F_2^2)\cos\alpha)}{\sqrt{(F_1^2+F_2^2+2F_1F_2\cos\alpha)(F_1^2+F_2^2+2F_1F_2\cos\alpha)}}$$

$$\frac{(1-\tan^2(\theta/2))}{(1+\tan^2(\theta/2))}=\frac{[2\,F_1\,F_2+(F_1^2+F_2^2)\cos\alpha]}{[(F_1^2+F_2^2)+2\,F_1\,F_2\cos\alpha]}$$

Thus , by componendo and dividendo,

$$=\frac{(F_1-F_2)^2-(F_1-F_2)^2\cos\alpha}{(F_1+F_2)^2+(F_1+F_2)^2\cos\alpha}$$

$$=\left(\frac{F_1-F_2}{F_1+F_2.}\right)^2\frac{1-\cos\alpha}{1+\cos\alpha}=\left(\frac{F_1-F_2}{F_1+F_2.}\right)^2\frac{2sin^2\frac{\alpha}{2}}{2\,cos2\frac{\alpha}{2}.}$$

$$=\left(\frac{F_1-F_2}{F_1+F_2.}\right)^2\tan^2\frac{\alpha}{2}$$

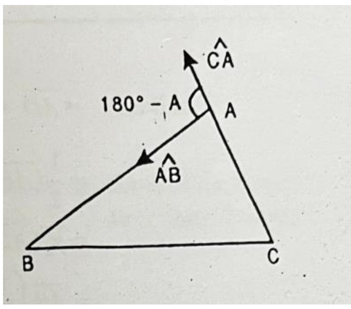
From which we get the result as stated.

Equilibrium: When the resultant of the forces acting at a point is zero, the orces are said to be in equilibrium.

Examples

Examples 1: Forces of magnitude F_1,F_2, F_3act on a particle.If their directions are parallel to $\overline{BC}$,$\overline{CA}$, $\overline{AB}$, Where ABC is a triangle,show that the magnitude of their resultant is $\sqrt{F_1{}^2+F_2{}^2+F_3{}^2-2F_2F_3cosA-2F_3F_1cosB-2F_1F_2cosC}$.

Soln. :



The given forces are$F_1\widehat{BC}$,$F_2\widehat{CA}$, $F_3\widehat{AB}$

where

$\widehat{BC}$,$\widehat{CA}$, $\widehat{AB}$

are the unit vectors parallel to $\overline{BC}$,$\overline{CA}$, $\overline{AB}$.If the magnitude of their resultant is, then we have

$$F^2=(F_1\widehat{BC}+F_2\widehat{CA}+F_3\widehat{AB}).\ (F_1\widehat{BC}\ ,F_2\widehat{CA},\ F_3\widehat{AB})$$

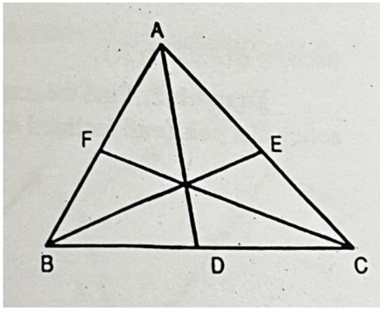
$$=\ F_1{}^2+F_2{}^2+F_3{}^2+2F_2F_3\widehat{CA}.\widehat{AB}+2\ F_3.\ F_1\widehat{AB}.\widehat{BC}+2\ F_1.F_2\widehat{BC}.\widehat{CA}$$

$$=\ F_1{}^2+F_2{}^2+F_3{}^2-2F_2F_3\cos A-2\ F_3.\ F_1\cos B-2\ F_1.F_2\cos C$$

Because $\widehat{CA}.\widehat{AB}$ =cos $(180° - A)$= -cos A etc

Example 2: The sides BC, CA,AB of a ΔABC are bisected inD,E,F.Show that the forces represented by DA, EB,FCare in equilibrium.

Soln. :



The forces act through the centroid, since D is the midpoint of BC, $\overline{AD}$=1/2($\overline{AB}$+$\overline{AC}$)

=1/2($\overline{AB}$-$\overline{CA}$)

But$\overline{DA}$= -$\overline{AD}$

=- 1/2($\overline{AB}$-$\overline{CA}$)

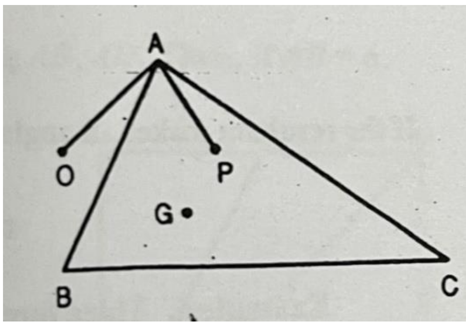
= -1/2($\overline{BC}$-$\overline{AB}$)

= -1/2(C$\bar{A}$-$\overline{BC}$)

Adding these three, we get the resultant as 0.So the forces are in equilibrium.

Example 3ABC is a triangle. G is its centroid and is any point in the plane of the triangle.

Show that the resultant of forces represented by$\overline{PA}\overline{PB}\overline{PC}$is 3$\overline{PG}$and find the position of P, if the three forces are in equilibrium.



Let O be any base point. Then it is evident that

$\overline{PA}=\overline{PB}+\overline{GO}+\overline{OA}$

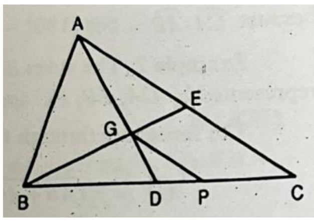
$\overline{PA} \;= 3\,\overline{PG}+3\,\overline{GO}+(\overline{OA}+\overline{OB}+\overline{OC})$

$=3\,\overline{PG}-3\overline{OG}+3(\overline{OG})$

$=3\,\overline{PG}$

This becomes zero and the forces are in equilibrium when P coincides with the centroid G.

Example 4:ABC is a triangle and D,E,F are the midpoints of the sides.Forces represented by AD ,2/3 BE and 1/3 CF act on a particle at the point where AD and BE meet.Show that the resultant is represented in magnitude and direction by ½AC and that its line of action divides BC in the ratio 2:1 .



Since D is the midpoint of BC,$\overline{AD}$ = ½ ($\overline{AB}$+$\overline{AC}$),etc.So

resultant = $\overline{AD}$ + 2/3 $\overline{BE}$ + 1/3 $\overline{CF}$

$=\frac{1}{2}(\overline{AB}+\overline{AC})+\frac{2}{3}.\frac{1}{2}(\overline{BA}+\overline{BC})+\frac{1}{3}.\frac{1}{2}(\overline{CA}+\overline{CB})$

$=\frac{1}{2}(\overline{AB}-\overline{CA})+\frac{1}{3}(-\overline{AB}+\overline{BC})+\frac{1}{6}(\overline{CA}-\overline{BC})$

$=\overline{AB}\,(\frac{1}{2}-\frac{1}{3})+\overline{BC}\,(\frac{1}{3}-\frac{1}{6})+\overline{CA}\,(-\frac{1}{2}+\frac{1}{6})$

$= 1/6\,[\overline{AB}+\overline{BC}+\overline{CA}] - 1/2\,\overline{CA}$

$= 1/6\,[\overline{0}] + 1/2\,\overline{AC}$

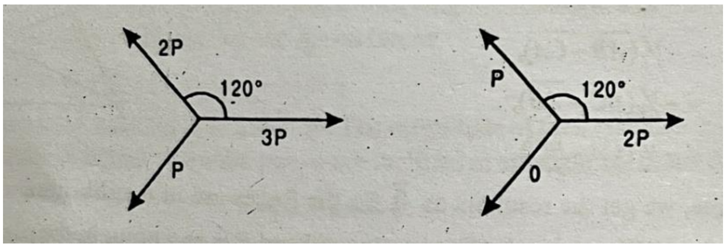
Thus ,the resultant is $1/2\ \overline{AC}$ parallel to the side AC and it passes through the centroid which is the meeting point of the medians AD, BE.

Let the resultant intersect BC at P. Then GP is parallel to AC and

$$\frac{BP}{PC}=\frac{BG}{GE}=\frac{2}{1}$$

because BG: GE = 2: 1.

Example 5. Find the magnitude and direction of the resultant of three coplanar forces P, 2P, 3P acting at a point and inclined mutually at an equal angle of 120°.



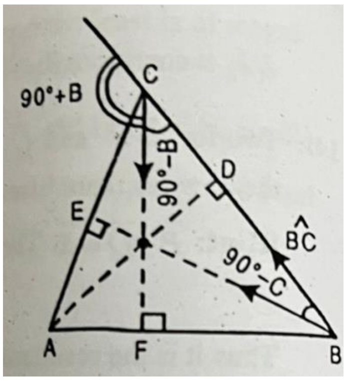
Considering only P, P, P, from the given forces we see that their resultant is zero since they are equally inclined to one another. So ,the remaining forces are 2P, P as in the second figure. Their resultant is

$\sqrt{(4P^2 + P^2 - 4P^2 \cos 120°)} = \sqrt{3}\ P.$

If the resultant makes an angle θ with the given force 3P, then

$\tan\theta = \frac{(P \sin 120°)}{(2P + P\cos 120°)} = \frac{1}{\sqrt{3}}$ or $\theta = 30°.$

Example 6. Three forces proportional to the sides of a triangle act at the vertices towards and perpendicular to the corresponding sides. Prove that the forces are in equilibrium.



Let ABC be the triangle and AD, BE, CF its altitudes. Then the given forces are of the form

$\lambda a\widehat{AD}, \lambda b\widehat{BE}, \lambda c\ \widehat{CF},$

These forces concur at the orthocentre. So their resultant acts through the orthocentre. This resultant is

λ (a $\widehat{AD}$, b $\widehat{BE}$, c $\widehat{CF}$,). (1)

Multiplying this scalarly by $\widehat{BC}$, we get

$\lambda\{a(0) + b\cos(90° - C) + c\cos(90° + B)\}$

or$\lambda(b\sin C - c\sin B)$.

This quantity vanishes because we have

$$\frac{b}{sinB} = \frac{c}{sinC}$$

So the vector (1) either vanishes or is perpendicular to BC. Similarly, it either vanishes or is perpendicular to CA. It cannot be perpendicular to both BC and CA. So it must vanish.

Remark. Three forces proportional to the sides of a triangle acting perpendicular to the sides at their midpoints, are in equilibrium.

In this case the forces meet at the circumcentre S. If D, E, F are the midpoints of the sides, the resultant is

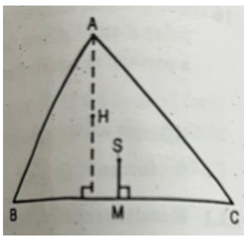
$\lambda(a\widehat{DS}+\lambda b\widehat{ES}+\lambda c\widehat{FS})$.

Multiply this scalarly by BC, etc.

Example 7. S and H are the circumcentre and orthocentre of a triangle ABC. Show that

(i) the resultant of the forces $\overline{SA}$,$\overline{SB}$, $\overline{SC}$, acting at S is $\overline{SH}$,

(ii) the resultant of the forces$\overline{HA}$,$\overline{HB}$, $\overline{HC}$ acting at H is $2\overline{HS}$.



Let M be the midpoint of BC. Then M divides BC in the ratio 1:1. So

$\overline{SM}=\frac{\overline{SB}+\overline{SC}}{2}$

$\overline{SB}+\overline{SC}=2\overline{SA}$

Similarly, we have

$\overline{HA}+\overline{HC}=2\overline{HM}$

But we know, from geometry, that AH is parallel to SM and AH = 2SM. So

$\overline{AH} = 2\overline{SM}$

(i) $\overline{SA} + \overline{SB} + \overline{SC} = \overline{SA} + 2\overline{SM}$

$= \overline{SA} + \overline{AH}$

$= \overline{SH}$

(ii) $\overline{HA} + \overline{HB} + \overline{HC} = \overline{HA} + 2\overline{HM}$

$= 2\overline{MS} + 2\overline{HM}$

$= 2(\overline{MS} + \overline{HM})$

$= 2(\overline{HM} + \overline{MS})$

$= 2\overline{HS}$

1.5 Resultant of several forces acting on a particle :

Bookmark 1.4 . To find the resultant of coplanar forces using their components .

Let us find the resultant of the forces $\bar{F}_1$, $\bar{F}_2$, …….,$\bar{F}n$.Now the resultant is

$\bar{F}_1 + \bar{F}_2 + \ldots\ldots + \bar{F}n = X\bar{\imath} + Y\bar{\jmath}$, say

Multiplying this scalarly by **î**,

$$\overrightarrow{F_1} \cdot \hat{\imath} + \overrightarrow{F_2} \cdot \hat{\imath} + \cdots + \overrightarrow{F_n} \cdot \hat{\imath} = X\hat{\imath} \cdot \hat{\imath} = X$$

Now$\overrightarrow{F_1} \cdot \hat{\imath}$is the component of $\overrightarrow{F_1}$in the î direction.So X is the sum of the components of the forces in the î direction.Similarly Y is the sum of the components of the forces in the ĵ direction.Now the magnitude of the resultant is

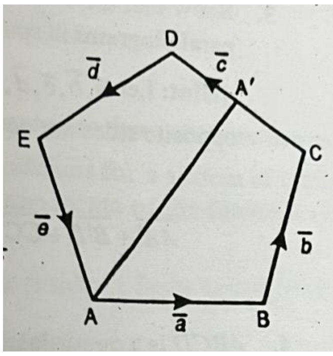
$$| X\hat{\imath} + Y\hat{\jmath} | = \sqrt{X^2 + Y^2}$$

and the angle between the resultant and the **î** direction is

$$\tan^{-1}\left(\frac{Y}{X}\right)$$

EXAMPLES

Example 1: Five forces acting at a point are represented in magnitude and direction by the lines joining the vertices of any pentagon to the midpoints of their opposite sides. Show that they are in equilibrium.



Let ABCDE be the pentagon and A′, B′, C′, D′, E′ be the midpoints of sides opposite to A, B, C, D, E.Let

$$\overrightarrow{AB} = \vec{a}, \overrightarrow{BC} = \vec{b}, \overrightarrow{CD} = \vec{c}, \overrightarrow{DE} = \vec{d}, \overrightarrow{EA} = \vec{e}$$

Then

$$\vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e} = 0$$

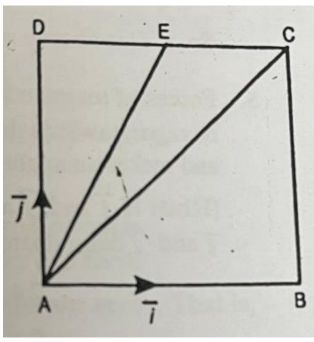
Now considering the force $\overrightarrow{AA'}$, we have

$$\overrightarrow{AA'} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{a} + \vec{b} + \frac{1}{2}\vec{c}$$

$$\sum\overrightarrow{AA'} = \frac{1}{2}(\vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e}) = 0$$

Hence the forces are in equilibrium.

Example 2: E is the midpoint of the side CD of a square ABCD.Forces 16, 20, 4√5, 12√2 act along AB, AD, EA, CA.Show that they are in equilibrium.



All the forces act through A.Let î, ĵ be the unit vectors along AB, AD.If AB = a, then

$$\overrightarrow{EA} = (-\frac{1}{2}\hat{\imath} - \hat{\jmath})a$$

$$\overrightarrow{CA} = (-\hat{\imath} - \hat{\jmath})a$$

So the unit vectors along EA, CA are

$$\frac{-\frac{1}{2}\hat{\imath} - \hat{\jmath}}{\sqrt{\frac{1}{4} + 1}}, \frac{-\hat{\imath} - \hat{\jmath}}{\sqrt{2}}$$

Resultant$= 16\hat{\imath} + 20\hat{\jmath} + 4\sqrt{5}\left(\frac{-\frac{1}{2}\hat{\imath} - \hat{\jmath}}{\sqrt{\frac{1}{4}+1}}\right) + 12\sqrt{2}\left(\frac{-\hat{\imath}-\hat{\jmath}}{\sqrt{2}}\right)$

$$= 16\hat{\imath} + 20\hat{\jmath} - 4\sqrt{5}\left(\frac{2}{\sqrt{5}}\right)(\frac{1}{2}\hat{\imath} + \hat{\jmath}) - 12(\hat{\imath} + \hat{\jmath})$$

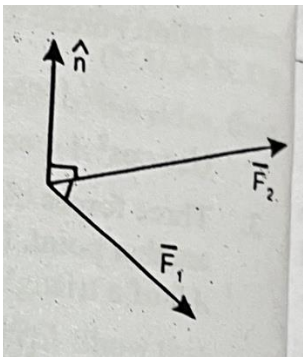
$$= \hat{\imath}(16 - 4 - 12) + \hat{\jmath}(20 - 8 - 12) = 0$$

Hence the forces are in equilibrium.

1.6 EQUILIBRIUM OF A PARTICLE

1.6.1 Equilibrium of a Particle

When the resultant of the forces acting at a point is zero, then the forces are said to be in equilibrium. In this case the particle is at rest in spite of the forces.



1.6.2 Equilibrium of a Particle under Three Forces

First we consider cases in which a particle is in equilibrium under the action of three forces.

Bookwork 1To show that, if three forces keep a particle in equilibrium, then the forces are coplanar.

Let the forces $\overline{F_1}, \overline{F_2}, \overline{F_3}$ keep a particle in equilibrium. Then the resultant force on the particle is $\overline{F}_1 + \overline{F}_2 + \overline{F}_3$

Since the particle is in equilibrium,

$$\overline{F}_1 + \overline{F}_2 + \overline{F}_3 = 0$$

Let $\hat{\mathbf{n}}$ be the unit vector perpendicular to both $\overline{F}_1$ and $\overline{F}_2$. Then

$\hat{n}.\,\overline{F}_1 = 0$ $\qquad\qquad\qquad$ $\hat{n}.\,\overline{F}_2 = 0$

Now, multiplying (1) scalarly by $\hat{n}$, we see that

$\hat{n}.\,(\overline{F}_1 + \overline{F}_2 + \overline{F}_3) = 0$ $\qquad$ or $\hat{n}.\overline{F}_1 + \hat{n}.\,\overline{F}_2 + \hat{n}.\overline{F}_3 = 0$

But $\hat{n}.\ \overline{F}_1 = 0$ and $\hat{n}.\ \overline{F}_2 = 0$ .So $\hat{n}.\ \overline{F}_3 = 0$ which means that $\overline{F}_3$ is also perpendicular to $\hat{n}$.

Therefore, $\overline{F_1}, \overline{F_2}$ *and* $\overline{F_3}$ are coplanar.

Bookwork 2 (Triangle of Forces) If three forces acting on a particle can be represented in magnitude and direction by the sides of a triangle, taken in order, then the forces keep the particle in equilibrium.

Let the given forces be represented in magnitude and direction by the sides AB, BC, CA of a triangle ABC. Then the forces are

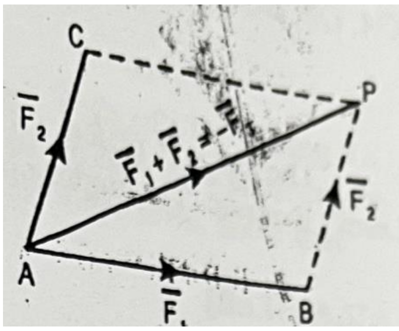
$$\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$$

Their resultant is

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$But by vector theory,

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \bar{0}$Hence the resultant force acting on the particle is zero. So the particle is in equilibrium.

Bookwork 3 (Converse of Triangle of Forces) If a particle is kept in equilibrium by three forces, then the forces can be represented in magnitude and direction by the sides of a triangle, taken in order.



Let the given forces be $\overline{F_1}, \overline{F_2}, \overline{F_3}$.They keep the particle in equilibrium. So their resultant is zero. Hence,

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$$

or

$$\vec{F}_1 + \vec{F}_2 = -\vec{F}_3$$

Let AB, AC denote $\vec{F}_1$in magnitude and direction.Complete the parallelogram ABPC. Then BP denotes $\vec{F}_2$and AP denotes $\vec{F}_1 + \vec{F}_2$in magnitude and direction.But$\vec{F}_1 + \vec{F}_2 = -\vec{F}_3$.So AP denotes $-\vec{F}_3$. Hence PA denotes $\vec{F}_3$.This completes the proof.

Polygon of Force: It can be easily seen that if several coplanar forces acting on a particle can be represented in magnitude and direction by the sides of a polygon, taken in order, the forces keep the particle in equilibrium.This result is called the **polygon of forces**.

Converse of Polygon of Forces: The converse of polygon of forces is that: If a particle is kept in equilibrium by forces, then they can be represented by the sides of an n-sided polygon.To prove the truth of this, let us consider, in particular, six forces$\vec{F}_1, \vec{F}_2, \vec{F}_3, \vec{F}_4, \vec{F}_5, \vec{F}_6$which keep a particle in equilibrium.Because of equilibrium,

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6 = 0 \qquad (1)$$

Take points A, B, C, D, E, F such that

$$\overrightarrow{AB} = \vec{F}_1,\ \overrightarrow{BC} = \vec{F}_2,\ \overrightarrow{CD} = \vec{F}_3,\ \overrightarrow{DE} = \vec{F}_4,\ \overrightarrow{EF} = \vec{F}_5$$

Now, from vector theory, we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = 0 \qquad (2)$$

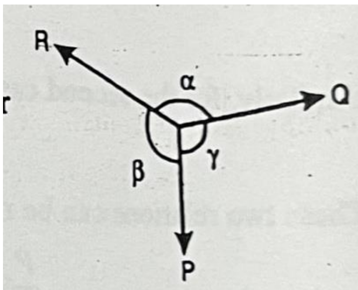
From (1) and (2), we get$\overrightarrow{FA} = \vec{F}_6$This completes the proof.

Lami's Theorem: The following bookwork is an important one. This is called **Lami's theorem**.

Bookwork 4: If a particle is in equilibrium under the action of three forces P, Q, R, then show that

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Where$\alpha$is the angle between Q and R,$\beta$is the angle between R and P,$\gamma$is the angle between and Q,and$|\ P\ | = P$, etc.



The forces keep the particle in equilibrium. So they are coplanar and their resultant is zero. Hence,

$$\vec{P} + \vec{Q} + \vec{R} = 0$$

Multiplying this vectorially by $\vec{P}$,

$$\vec{P} \times \vec{P} + \vec{P} \times \vec{Q} + \vec{P} \times \vec{R} = 0 \qquad (1)$$

Let $\hat{n}$be the unit vector perpendicular to the forces such that$\vec{P}, \vec{Q}, \hat{n}$form a right-handed triad.Then (1) becomes

$$0 + PQ\sin\gamma\,\hat{n} + PR\sin\beta(-\hat{n}) = 0$$

$$(PQ\sin\gamma - PR\sin\beta)\hat{n} = 0$$

$$PQ\sin\gamma - PR\sin\beta\ = 0 \qquad or \qquad PQ\sin\gamma = PR\sin\beta$$

$$\frac{Q}{\sin\beta} = \frac{R}{\sin\gamma} \qquad (2)$$

Similarly, the vectorial multiplication of$\vec{P} + \vec{Q} + \vec{R} = 0$ by $\vec{Q}$gives

$$\frac{P}{\sin\alpha} = \frac{R}{\sin\gamma} \qquad (3)$$

From (2) and (3), we get

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Remark: If four forces $F_1\hat{e}_1, F_2\hat{e}_2, F_3\hat{e}_3, F_4\hat{e}_4$acting on a particle keep it in equilibrium, where$\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4$are unit vectors, then

$$\frac{F_1}{[\hat{e}_2\hat{e}_3\hat{e}_4]} = -\frac{F_2}{[\hat{e}_1\hat{e}_3\hat{e}_4]} = \frac{F_3}{[\hat{e}_1\hat{e}_2\hat{e}_4]} = -\frac{F_4}{[\hat{e}_1\hat{e}_2\hat{e}_3]}$$

because multiplying $F_1\hat{e}_1 + F_2\hat{e}_2 + F_3\hat{e}_3 + F_4\hat{e}_4$=0 scalarly by $\hat{e}_3 \times \hat{e}_4$, we get

$F_1[\hat{e}_1\hat{e}_3\hat{e}_4]$+$F_2[\hat{e}_2\hat{e}_3\hat{e}_4]$=0 or$\frac{F_1}{[\hat{e}_2\hat{e}_3\hat{e}_4]} = -\frac{F_2}{[\hat{e}_1\hat{e}_3\hat{e}_4]}$

1.6.3 Equilibrium of a Particle under Several Forces

Now we study the situation in which a particle is in equilibrium under three or more forces.

Bookwork 5: To prove that the necessary and sufficient conditions for a system of coplanar forces to keep a particle in equilibrium is that the sums of the components of the forces in two mutually perpendicular directions in the plane are zero.

Necessity Part: Let the forces be$\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n$.Then the resultant force acting on the particle is

$$\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n$$

So, if the particle is in equilibrium, then

$$\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n = 0 \qquad (1)$$

Let **î, ĵ** be the unit vectors in two perpendicular directions.Multiplying (1) scalarly by **î** and **ĵ**,

$$(\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n)\cdot\hat{\imath} = 0, (\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n)\cdot\hat{\jmath} = 0$$

or$\vec{F}_1\cdot\hat{\imath} + \vec{F}_2\cdot\hat{\imath} + \cdots + \vec{F}_n\cdot\hat{\imath} = 0 \vec{F}_1\cdot\hat{\jmath} + \vec{F}_2\cdot\hat{\jmath} + \cdots + \vec{F}_n\cdot\hat{\jmath} = 0$

This proves the necessity part that:

If the particle is in equilibrium, then the sums of the components of the forces in two mutually perpendicular directions are *zero.*

Sufficiency Part: Now we have that the sums of the components are zero. That is,

$$\vec{F}_1\cdot\hat{\imath} + \vec{F}_2\cdot\hat{\imath} + \cdots + \vec{F}_n\cdot\hat{\imath} = 0$$

$$\vec{F}_1 \cdot \hat{j} + \vec{F}_2 \cdot \hat{j} + \cdots + \vec{F}_n \cdot \hat{j} = 0$$

These imply that

$\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n) \cdot \hat{\imath} = 0, (\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n) \cdot \hat{j} = 0$

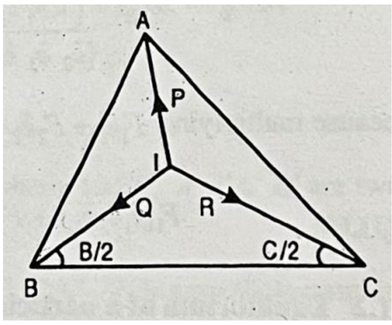
That is, the resultant force is either perpendicular to both i and j or is a zero force. The perpendicularity cannot happen. Thus

$$(\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n) = 0$$

EXAMPLES

Example 1. I is the incentre of a triangle ABC. If forces of magnitudes P, Q, R acting along the bisectors IA, IB, IC are in equilibrium, show that

$\frac{P}{\cos(A/2)} = \frac{Q}{\cos(B/2)} = \frac{R}{\cos(C/2)}.$



The forces P, Q, R act at I and are in equilibrium. So we shall use Lami's theorem. The angles opposite to P, Q, Rare∠BIC, ∠CIA,∠AIB

$\frac{\text{P}}{\text{sinBIC}} = \frac{\text{Q}}{\text{sinCIA}} = \frac{\text{R}}{\text{sinAIB}} \dots (1)$

Now, from ΔBIC,

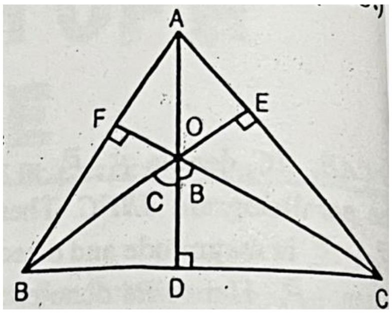
∠BIC = 180°− (B/2 + C/2)

sin BIC = sin(B/2 + C/2) = sin(90° − A/2) = cos(A/2)

Similarly, sin CIA = cosB/2, sin AIB = cosC/2.Thus (1) becomes

$\frac{P}{\cos(A/2)} = \frac{Q}{\cos(B/2)} = \frac{R}{\cos(C/2)}$

Example 2. I is the incentre of a triangle ABC. If the forces IA, IB, IC acting at I are in equilibrium, show that ABC is an equilateral triangle.



In the previous example we have the forces P, Q, R. Instead of them, now we have AI, BI, CI.

$$\frac{AI}{\cos(A/2)} = \frac{BI}{\cos(B/2)} = \frac{CI}{\cos(C/2)} \ldots (1)$$

But, if r is the radius of the incentre, then

$$\frac{\mathrm{r}}{\mathrm{AI}} = \sin\frac{\mathrm{A}}{2}, \frac{\mathrm{r}}{\mathrm{BI}} = \sin\frac{\mathrm{B}}{2}, \frac{\mathrm{r}}{\mathrm{CI}} = \sin\frac{\mathrm{C}}{2}$$

On eliminating AI, BI, CI from (1), we have

1 / (sin(A/2) cos(A/2)) = 1 / (sin(B/2) cos(B/2)) = 1 / (sin(C/2) cos(C/2))

i.e., $\frac{1}{\sin \mathrm{A}} = \frac{1}{\sin \mathrm{B}} = \frac{1}{\sin C}$

which implies that A=B=Cand so the triangle is equilateral.

Example 3. O is the orthocentre of triangle ABC. If forces of magnitude P, Q, R acting along OA, OB, OC are in equilibrium, show that

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$$

The forces P, Q, R act at O and are in equilibrium. So we shall use Lami's theorem. The angles opposite to P, Q, Rare ∠BOC, ∠COA, ∠AOB.

$$\frac{P}{\sin \mathrm{BOC}} = \frac{Q}{\sin \mathrm{COA}} = \frac{R}{\sin \mathrm{AOB}}.$$

If AD is the altitude through A,

∠BOD = C, ∠COD = B.

Since CBE = 90° − C, etc.

∠BOC = B + C.

$$\sin BOC = \sin(B + C) = \sin(180° - A) = \sin A.$$

Similarly, sin COA = sin B, sin AOB = sin C. Thus we get

$\frac{P}{\text{sinA}} = \frac{Q}{\text{sinB}} = \frac{R}{\text{sinC}}.$

But by sine formula, sin A: sin B: sin C = a: b: c.

$$\therefore \frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$$

Example 4. Three forces of magnitudes P, Q, R acting at a point being parallel to the sides of a triangle are in equilibrium. If another set of forces of magnitudes P′, Q′, R′ acting at a point parallel to the sides of the same triangle are also in equilibrium, show that

$\frac{P}{P'} = \frac{Q}{Q'} = \frac{R}{R'}$

Let ABC be the triangle. Then the forces acting at the first point are P$\widehat{BC}$, Q $\widehat{CA}$, R $\widehat{AB}$

. Since they are in equilibrium,

$$P\cdot\widehat{BC} + Q\cdot\widehat{CA} + R\cdot\widehat{AB} = 0.$$

Similarly, for the second case,

$$P'\cdot\widehat{BC} + Q'\cdot\widehat{CA} + R'\cdot\widehat{AB} = 0.$$

These two relations can be rewritten as

$\frac{P}{R}\widehat{BC} + \frac{Q}{R'}\widehat{CA} = -\text{AB}\,, \frac{P'}{R}\widehat{BC} + \frac{Q'}{R'}\widehat{CA} = \text{-AB}$

Eliminating AB by subtraction,

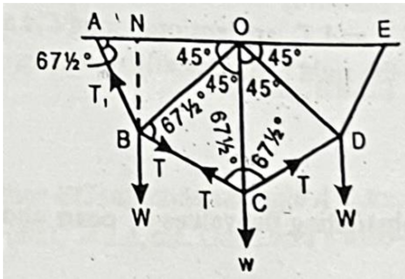
$(\frac{P}{R} -, \frac{P'}{R'})\widehat{BC} + (\frac{Q}{R} - \frac{Q'}{R'})\widehat{CA} = 0$

If a and b are two non-parallel vectors and l ·a + m ·b = 0, then l = 0 and m=0.So

$$\frac{P}{R} -, \frac{P'}{R'} = 0 \frac{Q}{R} - \frac{Q'}{R'} = 0$$

These leads to the result as stated.

Example 5. Weights W, w, W are attached to points B, C, D, respectively of a light string AE where B, C, D divide the string into 4 equal lengths. If the string hangs in the form of 4 consecutive sides of a regular octagon with the ends A and E attached to points on the same level, show thatW = (√2 + 1) w.



Let ABCDE be the lower half of the octagon. Each side of the octagon subtends at the centre O an angle

$\frac{180°}{4}$or 45 °

Now triangles OAB, OBC, OCD, ODE are isosceles triangles with vertical angles 45° and the base angles 67½°. The figure is self explanatory.

The forces at C are w and the equal tensions T, T (equal due to symmetry).The angles opposite to w and T are

∠BCD, ∠BCW or 67½° + 67½°, 180°– 67½°.

So, by Lami's theorem,

$\frac{w}{\sin(67½°+67½°)} = \frac{T}{\sin(180°-67½°)}$ or

$$\frac{w}{\sin 135°} = \frac{T}{\sin 67½°}$$

The forces at B are W, T, T_1. The angles opposite to W and T are

∠ABC, ∠ABW or 67½° + 67½°, 180°– 22½°

$$\frac{w}{sin 135°} = \frac{T}{sin 22½°}$$

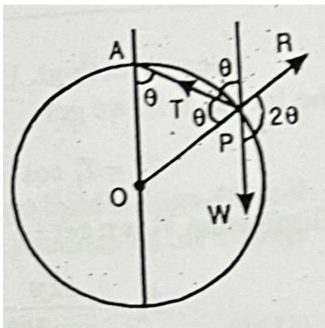
Dividing (2) by (1), we get

$\frac{w}{w} = \frac{\sin 67½°}{\sin 22½°} = \frac{\cos 22½°}{\sin 22½°} = \frac{1}{\tan 22½°}$

$$= \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{\sqrt{2}-1} = \sqrt{2} + 1$$

∴ W = (√2 + 1) w.

Example 6.A heavy bead of weight W can slide on a smooth circular wire in a vertical plane.The bead is attached to a light thread to the highest point of the wire, and in equilibrium,the thread is taut and makes an angle θ with the vertical. Show that the tension of thethread and the reaction of the wire on the bead are 2W cos θ and W.



Let A be the highest point of the circle, O the centre and P the bead.It is given that∠OAP= θ.But OA = OP and so ΔOAP is isosceles.Thus∠OPA = θ.Now the forces on the bead *are*

(i) Tension T

(ii) Normal Reaction R

(iii) Weight W

The angles opposite to them are

20°, 180° − θ, 180° − θ

Therefore, by Lami's theorem, we have

$$\frac{T}{\sin 20°} = \frac{R}{\sin(180° - \theta)} = \frac{W}{\sin(180° - \theta)}$$

$\frac{T}{\sin 20°} = \frac{R}{\sin\theta} = \frac{W}{\sin\theta}$ or $\frac{T}{\cos\theta} = R = W$

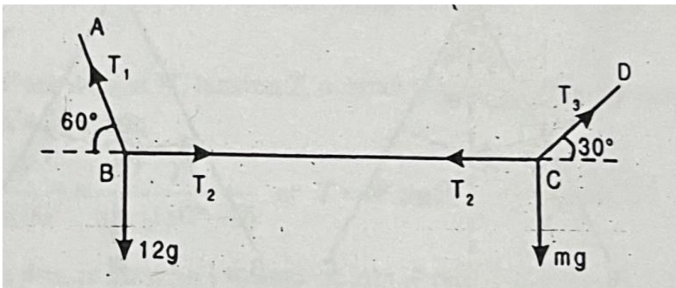
T=2Wcosθ ,R=W

Example 7. A bead of weight W is free to slide on a smooth vertical circle and is connected by a string whose length equals the radius of the circle to the highest point of the circle. Find the tension of the string and the reaction of the circle.

In the previous example AP was not given. But now AP = radius. So $\triangle$OAP is equilateral and thus $\theta = 60°$. Therefore

$$\frac{T}{\sin 120°} = \frac{R}{\sin 120°} = \frac{W}{\sin 120°} \text{or} \quad T = R = W.$$

Example 8. A string ABCD hangs from fixed points A, D carrying a mass of 12 kg at B and a mass of m kg at C. AB is inclined at 60° to the horizontal, BC is horizontal and CD is inclined at 30° to the horizontal. Show that m = 4.



First method. Let the forces acting on the system be T1, T2, T3, 12g, mg as shown in the figure. By Lami's theorem, for the forces at B,

$$\frac{T2}{\sin(90°-60°)} = \frac{12g}{\sin(180°-60°)} \text{or} \quad T_2 = \frac{12g}{\tan 60°}$$

But, for the forces at C, we have

$$\frac{T2}{\sin(90°+60°)} = \frac{mg}{\sin(180°-30°)} \quad \text{or } T_2 = \frac{mg}{\tan 30°}$$

$$\therefore \frac{m}{\tan 30°} = \frac{12}{\tan 60°} \quad \text{or } m = \frac{12}{\sqrt{3}\sqrt{3}}$$

Second method: Using the same figure, considering the horizontal and vertical components for the forces at B, we get

$$T_2 = T_1 \cos 60°, T_1 \cos 30° = 12g$$

Eliminating T_1, we get

$$T_2 = \frac{12g}{\cos 30°} \cos 60°$$

Similarly, for the forces at **C**, we have

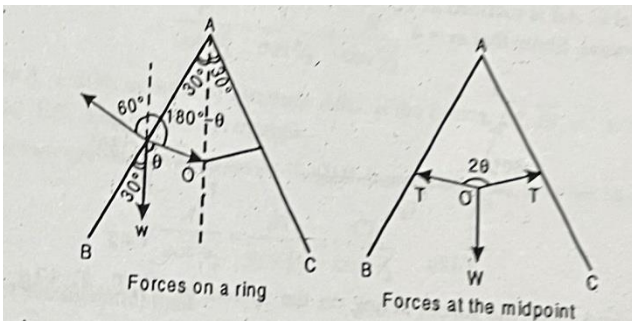
$$T_2 = T_3 \cos 30°, T_3 \cos 60° = mg$$

$$\therefore T_2 = \frac{mg \cos 30°}{\cos 60°}$$

Equating the values of T_2, we get m=4

Example 9 Two fixed smooth bars AB, AC in a vertical plane are each inclined at $30°$to the vertical.The ends of a light string are tied to two rings each of weight $w$which slide on the bars.From the midpoint of the string is hung a weight W.Prove that, in the position of equilibrium, each half of the string will make an angle $\theta$with the vertical given by

$$\tan\theta = \frac{W + 2w}{W}\sqrt{3}$$



The forces acting on each ring are tension Tinclined to the vertical at an angle θ, normal reaction R perpendicular to the rod, and weight w.Resolving along the rod downward,

$$w\cos 30° + T\cos(\theta + 30°) = 0 \qquad \dots(1)$$

The forces acting at the midpoint of the string are the tensions $T$and $T$and the weight W.Resolving them vertically downward,

$$W - 2T\cos\theta = 0 \qquad \dots(2)$$

Eliminating T, we get

$$w\cos 30°(2\cos\theta) + W\cos(\theta + 30°) = 0$$

or

$$2w\cos 30°\cos\theta + W(\cos\theta\cos 30° - \sin\theta\sin 30°) = 0$$

or

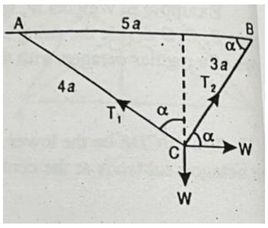
$$2w + W(1 - \tan\theta\tan 30°) = 0$$

or

$$W\tan\theta\tan 30° = 2w + W$$

$$\therefore \tan\theta = \frac{2w + W}{W}\sqrt{3}$$

Example 10: A particle C of weight W is in equilibrium being supported by two strings CA, CB of length 4a, 3a respectively and acted on by a horizontal force W in the plane ABC. If the ends A, B are at the same level and at a distance 5a apart, show that the tensions in the strings are $\frac{7W}{5}, \frac{W}{5}$.



Since $(4a)^2 + (3a)^2 = (5a)^2$,

$$\angle ACB = 90°$$

If CA is inclined to the vertical at an angle α, then

$$\cos\alpha = \frac{3}{5}, \sin\alpha = \frac{4}{5}$$

If T_1 and T_2 are tensions along CA and CB, resolving horizontally to the right and vertically upwards,

$$T_2\cos\alpha - T_1\sin\alpha + W = 0$$

$$T_2\sin\alpha + T_1\cos\alpha - W = 0$$

Substituting the values of $\cos\alpha$ and $\sin\alpha$ and simplifying,

$$3T_2 - 4T_1 + 5W = 0$$

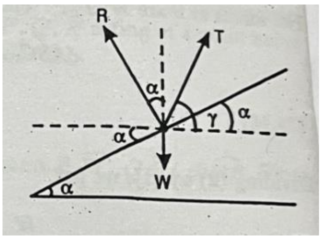
$$4T_2 + 3T_1 - 5W = 0$$

Solving these, we get

$$T_1 = \frac{7W}{5}, T_2 = \frac{W}{5}$$

Example 11: A weight is supported on a smooth plane of inclination α by a string inclined to the horizon at an angle γ. If the slope of the plane be increased to β and the slope of the string be unaltered, the tension of the string is doubled. Prove that

$$\cot\alpha - 2\cot\beta = \tan\gamma$$



The forces acting on the particle are:

i. Normal reaction R
ii. Weight W
iii. Tension T

Considering the vertical components, we have

$$R\cos\alpha + T\sin\gamma = W$$

or

$$R\cos\alpha = W - T\sin\gamma \ldots (1)$$

From the horizontal components, we get

$$R\sin\alpha = T\cos\gamma \ldots (2)$$

Dividing (1) by (2), we get

$$\cot\alpha = \frac{W}{T\cos\gamma} - \tan\gamma \ldots (3)$$

Whenαbecomes β, Tbecomes $2T$.So,

$$\cot\beta = \frac{W}{2T\cos\gamma} - \tan\gamma \ldots (4)$$

$$(3) - 2(4) \Rightarrow \cot\alpha - 2\cot\beta = \tan\gamma$$

1.7 LIMITING EQUILIBRIUM OF A PARTICLE ON AN INCLINED PLANE

Bookwork 6. Suppose a particle of weight W lying on a rough plane inclined at an angle α to the horizontalis subjected to a force P along the plane in the upward direction.If the equilibrium is limiting, to find P.

Now the following two cases of limiting equilibrium arise:

(i) The particle has a tendency to slide up the plane.

(ii) The particle has a tendency to slide down the plane.

Case (i) Now it is enough to consider the Forces acting along the particle along the plane: They are

(i) Component of weight W sin α down the plane

(ii) P up the plane

(iii) Frictional force F down the plane

where F has its maximum value so that

$F = \mu R$ or $F = \mu(W \cos \alpha)$

Where μ is the coefficient of friction Thus, in this case:

$P = W \sin \alpha + F$ or $P = W \sin \alpha + \mu(W \cos \alpha)$.

Case (ii) In the case in which the particle has a tendency to move downward,the frictional force is up the plane and hence we have:

$W \sin \alpha = P + \mu W \cos \alpha$ or $P = W (\sin \alpha - \mu \cos \alpha)$.

Bookwork 7. Suppose a particle of mass m is placed on a rough inclined plane inclined at an angle α to the horizontal and a force of magnitude S acts on it in a direction making an angle θ with the plane. If the equilibrium is limiting, to find S.

Now the following two cases of limiting equilibrium arise:

(i) The particle has a tendency to slide up the plane.

(ii) The particle has a tendency to slide down the plane.

Case (i): Let the particle have the tendency to move up the plane.In this case the forces acting on the particle are:

(i) S, the applied force

(ii) R, the normal reaction normal to the plane

(iii) μR, the frictional force down the plane

(iv) mg, the weight

where μ is the coefficient of friction which is equal to the tangent of the angle of friction λ.That is,

$\mu = \tan\lambda$.

Now considering the components of these forces along the plane and normal to the plane,both in the upward sense,

$S\cos\theta - \mu R - mg\sin\alpha = 0$

$S\sin\theta + R - mg\cos\alpha = 0$

Elimination of R gives:

$S(\cos\theta + \mu\sin\theta) - mg(\sin\alpha + \mu\cos\alpha) = 0$

$\therefore S = \frac{mg(\sin\alpha + \mu\cos\alpha)}{(\cos\theta + \mu\sin\theta)}$

$= \frac{mg(\sin\alpha + \tan\lambda\cos\alpha)}{(\cos\theta + \tan\lambda\sin\theta)}$

$= \frac{mg(\sin\alpha\cos\lambda + \sin\lambda\cos\alpha)}{(\cos\theta\cos\lambda + \sin\lambda\sin\theta)}$

$= \frac{mg\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}$

If θ is unaltered and S is increased, then the particle will slide up the plane.

Minimum S: In the above expression, m, α and λ are constants.So S is a minimum when the denominator $\cos(\theta - \lambda)$ is a maximum,i.e., when $\theta - \lambda = 0$ or $\theta = \lambda$.Hence the force required to drag the particle up the plane will be the leastwhen it is applied in a direction making an angle equal to the angle of frictionwith the inclined plane.

Case (ii): Let the particle have a tendency to move down the plane.In this case the frictional force μR is in the upward sense.Here the corresponding value of S can be obtained directlyfrom the above working by simply changing μ into $-\mu$.Thus we get:

$S = \frac{mg\sin(\alpha - \lambda)}{\cos(\theta + \lambda)}$

Remark:In the above two cases the involved forces are four in number.If, however, instead of R andμR, their resultant, namely the reaction, is considered,then there will be only three forces and so Lami's theorem can be used to find S.The reaction is inclincedto the normal at an angle λ.

Examples

Example 1: A body of weight 4 Kg rest in limiting equilibrium on a rough plane whose slope is 30°.If the plane is rise to a slope of 60°.Find the force along the plane required to support the body.

Solution

In the first case the body has a tendency to slide down, so the force on the body along the plane are:

i) Component of weight = 4 sin 30° down the plane.

ii) Limiting friction μR, i.e., $\mu \times 4 \cos 30°$ up the plane.

$4 \sin 30° = \mu \times 4 \cos 30°$

$\mu = \sin 30° / \cos 30° = \tan 30° = 1 / \sqrt{3}$

When the inclination of the plane is increased to 60°,also the body has a tendency to slide down,so the force along the plane are:

i) Component of weight down the plane = 4 sin 60°

ii) Limiting friction $\mu \times 4 \cos 60°$ up the plane.

iii) Supporting force P up the plane

$$P=4\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{3}}+\frac{1}{2}$$

$$=2\sqrt{3}-\frac{2}{\sqrt{3}}$$

$$=\frac{6-2}{\sqrt{3}}=\frac{4}{\sqrt{3}}$$

Example 2 A particle rest on a plane inclined at 45° to the horizontal, being supported by a string along the line of greatest slope. If the ratio of the maximum and minimum tensions consistent with equilibrium is 2 : 1, find the coefficient of the friction.

When the body has tendencies to move down and up.

Let the tensions be T_1, T_2 $\therefore$ $T_2 = 2T_1$

Hence $w \sin 45° = \mu (w\ 4 \cos 45°) + T_1$..........................(1)

Hence $w \sin 45° + \mu (w\ 4 \cos 45°) - T_2 = 2T_1$ ....(2)

Where W is the weight of the particle and μ is the coefficient of friction.

from 1 $\quad T_1 = w \sin 45° - \mu (w\ 4 \cos 45°)$

$w \sin 45° + \mu w \cos 45° = 2(w \sin 45° - \mu w \cos 45°)$

$\mu w \cos 45° + 2 \mu w \cos 45° = 2 w \sin 45° - w \sin 45°$

$$3 \mu w \cos 45° = w \sin 45°$$

$\mu = \frac{\sin 45°}{3\cos 45°} = \frac{1}{3} \tan 45°$

$= \frac{1}{3}$

Example 3: A particle is placed on a rough plane whose inclination to the horizon is α and is acted on by a force P, parallel to the plane and in a direction making an angle β with the line of greatest slope in the plane. If the coefficient of friction is μ and the equilibrium is limiting, find the direction in which the body will begin to move.

Leaving out the normal reaction, we have along the plane forces:

i) P along OB say

ii) Component of weight W sin α down the plane along OA

iii) Limiting friction.

The angle between the first two is $180° - \beta$.

So the resultant of these two is along OC inclined at OA at an angle θ, namely.

$\Theta = \tan^{-1} \frac{P \sin(180° - \beta)}{W\sin\alpha + P\cos(180° - \beta)}$

$$\Theta = \tan^{-1} \frac{P \sin\beta}{W\sin\alpha + P\cos\beta}$$

The third force which is the limiting friction will be opposite to the direction.

So the body will have a tendency to move in the direction OC

Example 4: Find the least force required to drag a particle on a rough horizontal plane and show that the least force acts in a direction making with the horizontal an angle equal to the angle of friction.

Let a force of magnitude F be applied to the particle in the direction which makes an angle $\Theta$with the horizontal. If this force is such that the particle is in the limiting equilibrium then the other forces acting on the particle are,

i) The weight mg vertically downward

ii) the reaction P which makes an angle $\lambda$with the normal, where $\lambda$is the angle of friction.

Now,by Lami's theorem.

$$\frac{F}{\sin(180°-\lambda)} = \frac{mg}{sin(90°-\theta+\lambda)} = \frac{P}{sin(90°+\theta)}$$

$$F = \frac{\sin(180°-\lambda)}{sin(90°-\theta+\lambda)} mg$$

$$= \frac{mg\sin\lambda}{cos(-\theta+\lambda)}$$

Here $\lambda$is a constant . Hence F is a minimum when $\cos(-\theta + \lambda)$is a maximum

i.e, the minimum F is obtained from $\lambda - \theta$=0 or λ=0 which is the angle of friction.

EXERCISES

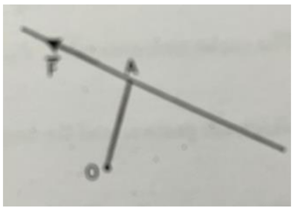
1. Two like parallel forces of magnitude P, Q act on a rigid body. If the second force is moved away from the first parallelly through a distance d, show that the resultant of the force moves through a distance $\frac{dQ}{P+Q}$.
2. Find the magnitude and the line of action of the resultant of parallel forces of magnitude 3, 6, 8 in one direction of magnitude 12 in the opposite direction acting at the points A, B, C, D in the straight line where AB=10, BC=30 and CD=50.

UNIT II

FORCES ON A RIGID BODY

2.1 MOMENT OF A FORCE

Moment of a force.



Let $\bar{F}$ be a force and A, a point on its line of action. Let O be a point in space. Then the vector

$$\overline{OA} \times \bar{F}$$

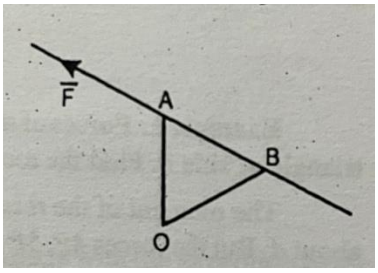
is called the moment of $\bar{F}$ about O.

Remark 1. The above moment is independent of the position of A on the straight line because, if B is any other point on the line, then

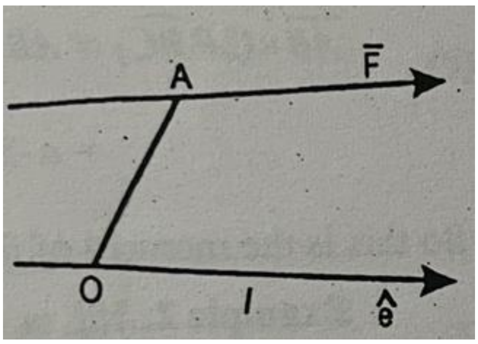
$$\overline{OB} \times \bar{F} = (\overline{OA} + \overline{AB}) \times \bar{F}$$

Remark 2. If the moment of $\bar{F}$ about A is zero, then either

i. $\bar{F} = \bar{0}$ or
ii. The line of action of $\bar{F}$ passes through A.



Moment of a force about a line

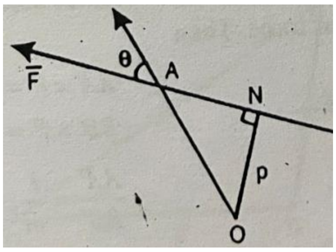


Let $\bar{F}$ be a force and A, a point on its line of action. Let l be a directed line through a point O, the direction of the line being specified by $\hat{e}$. Then the scalar triple product

$$(\overline{OA} \times \bar{F}) \cdot \hat{e}$$

is called the moment of the force $\bar{F}$ about l.

Scalar moment



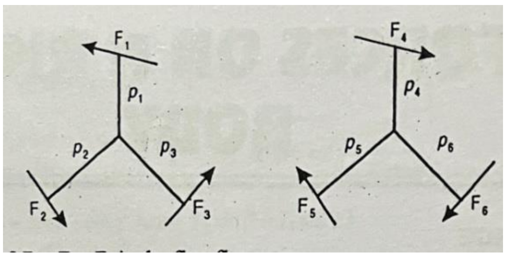
Let $\bar{F}$ be a force in a plane. Let A be a point on its line of action and O, any point in the plane. Let ON be the perpendicular from O to the line and

$$ON = p.$$

Then the moment of $\bar{F}$ about O is

$$\overline{OA} \times \bar{F} = OA \cdot F \cdot \sin\theta \cdot \hat{n} = pF\hat{n},$$

where θ is the angle between $\overline{OA}$ and $\bar{F}$, and $\hat{n}$ is the unit vector perpendicular to $\overline{OA}$, $\bar{F}$ such that $\overline{OA}$, $\bar{F}$, $\hat{n}$ form a right-handed triad. Now we call pF the scalar moment of $\bar{F}$ about O.



The scalar moments of F_1, F_2, F_3 in the first figure are

$$p_1F_1, p_2F_2, p_3F_3$$

which are positive, and the moments of F_4, F_5, F_6 in the second figure are

$$-p_4F_4, -p_5F_5, -p_6F_6$$

which are negative. The first three forces are such as to cause on a rigid body a rotational motion in the anticlockwise sense and the other three to cause a rotational motion in the clockwise sense.

EXAMPLES

Example 1. Forces of magnitudes $3P$,$4P$, $5P$, act along the sides BC, CA, AB of an equilateral triangle of side a. Find the moment of the resultant about A.

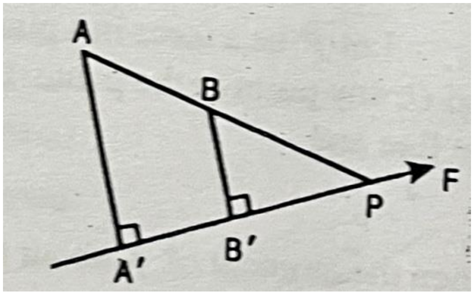
The moment of the resultant about A equals the sum of the moments of the individual forces about A. But the forces $4P$, $5P$ pass through A. Sotheir moments about A are zero. The moment of $3P$, which passes through B, is

$$\overline{AB} \times \left(3P\widehat{BC}\right) = AB \cdot 3P \cdot \sin 180°\,\hat{n}$$

$$= a \cdot 3P \cdot \frac{\sqrt{3}}{2}\hat{n}$$

So this is the moment of the resultant about A.

Example 2. If l, m, n are the moments of a force F in a plane about three non-collinear points A, B, C in the plane, show that the line of action of F divides AB in the ratio $l : m$. Deduce that, if l, m, n are given, then F can be found completely.



Suppose l, m, n are positive. Then A, B, C lie above the line of force if the force is from left to right. Let the line of force meet AB at P and A', B' be the feet of the perpendiculars from A, B to the line of force. Then

$$AA' \times F = l,$$

$$BB' \times F = m.$$

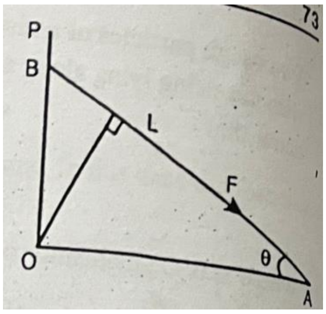
$$\therefore \frac{AA'}{BB'} = \frac{l}{m}. \text{ So } \frac{AP}{PB} = \frac{l}{m}.$$

That is, P is the point which divides AB externally in the ratio $l : m$.

Similarly, if Q is the point which divides BC externally in the ratio $m : n$, then $PQ$is the line of force.

Since $AA' \times F = l$, finding AA', we get Falso.

Example 3. One end of a rope of $20\ m$ is to be fixed to a telegraph post and the other end is to be pulled by a man on the ground with a constant force F. To cause the maximum effect to overturn the post, at what height the rope is to be fixed?



LetOP be the post, AB the rope and

$$\angle OAB = \theta.$$

Let OL be the perpendicular on AB. Now the post is to overturn about O. So the man should aim for the maximum moment of F about O. But the moment is$OL \times F$and it is a maximum when OL is a maximum. But

$$OL = OA \sin\theta, \text{ from } \Delta OAL$$
$$= (AB\cos\theta)\sin\theta, \text{ from } \Delta OAB$$
$$= {}^1\!/_2\, AB \sin 2\theta$$

which is a maximum when$2\theta = 90°$ or $\theta = 45°$. In this situation

$$OB = AB \sin 45° = 20 \times \frac{1}{\sqrt{2}}.$$

Note. Whatever be the length of the rope, θ should be $45°$.

2.2 GENERAL MOTION OF A RIGID BODY

In this section we extend Newton's laws of motion, $N.1, N.2, N.3$, to the motion of a rigid body.

Rigidbody.A system of particles such that the distance between any two of them is always constant, is called a rigid body.

Applied forces. Forces applied on a body by external agencies are called applied forces on the body.

Effective forces. If a particle of mass m has an acceleration$\ddot{\bar{r}}$, then the quantityis called the effective force of the particle.

With this nomenclature we have that the equation of motion of the particle, $m\ddot{\bar{r}} = \bar{F}$, is that

$$\begin{Bmatrix} \text{The effective force} \\ \text{on a particle} \end{Bmatrix} = \begin{Bmatrix} \text{The applied force} \\ \text{on the particle} \end{Bmatrix}$$

Equations of motion of a rigid body

The general motion of a rigid body is a composition of the motion of translation of the mass centre and the motion of rotation of the body about the mass centre. The equations of these two component motions are obtained in the following bookwork.

Bookwork 4.1. The bookwork has the following two parts:

(i) The mass centre of a rigid body of mass M moves as if it is a particle of mass M acted on by the applied forces on the rigid body, moved parallelly and made act on it.

(ii) The motion of a rigid body relative to the mass centre G is governed by the equation

$$\begin{Bmatrix} \text{Sum of the moments of} \\ \text{the effective forces, relative} \\ \text{to G, of the constituent} \\ \text{particles, about G} \end{Bmatrix} = \begin{Bmatrix} \text{Sum of the} \\ \text{moments of the} \\ \text{applied forces} \\ \text{about G} \end{Bmatrix}$$

Let us make the following assumptions:

$m_1, m_2, m_3 \cdots$	Masses of the particles constituting the rigid body
M	Total mass $m_1 + m_2 + m_3 + \cdots$
$\bar{r}_1, \bar{r}_2, \bar{r}_3 \cdots$	P.V's of $m_1, m_2, m_3 \cdots$ with reference to a fixed point O
$\bar{F}_{ij} (i \neq j)$	Force exerted on m_i by m_j due to their mutual contact
$\bar{F}_{ii}$	Applied force on m_i, including the forces due to externalconstraints
$\bar{F}$	$\bar{F}_{11} + \bar{F}_{22} + \bar{F}_{33} + \cdots$
$\bar{R}$	P.V's of the mass centre G of the rigid body, $\frac{m_1\bar{r}_1 + m_2\bar{r}_2 + \cdots}{M}$

Proof of (i). Now, by $N.3$, $\bar{F}_{ij}$ and $\bar{F}_{ji}$are equal in magnitude but opposite in direction.So,

$$\bar{F}_{ij} + \bar{F}_{ji} = \bar{0}. \qquad \ldots(1)$$

The equations of motion of $m_1,m_2,m_3 \cdots$ are

$$m_1\ddot{\bar{r}}_1 = \bar{F}_{11} + \bar{F}_{12} + \bar{F}_{13} + \cdots$$
$$m_2\ddot{\bar{r}}_2 = \bar{F}_{21} + \bar{F}_{22} + \bar{F}_{23} + \cdots$$
$$m_3\ddot{\bar{r}}_3 = \bar{F}_{31} + \bar{F}_{32} + \bar{F}_{33} + \cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

When these equations are added, using (1),

$$m_1\ddot{\bar{r}}_1 + m_2\ddot{\bar{r}}_2 + \cdots = \bar{F}_{11} + \bar{F}_{22} + \cdots = \bar{F}.$$

Thus we get that

$$\text{Sum of the effective forces} = \text{Sum of the applied forces.} \qquad \ldots(2)$$

But $m_1\bar{r}_1 + m_2\bar{r}_2 + \cdots = M\bar{R}$. Therefore

$$M\ddot{\bar{R}} = \bar{F}. \qquad \ldots(3)$$

Since $\ddot{\bar{R}}$ is the acceleration of the mass centre, we obtain the first result as stated.

Proof of (ii). Since $\bar{F}_{ij}$ and $\bar{F}_{ji}$are equal in magnitude but opposite in direction and act along the same line, their moments about O are equal in magnitude but opposite in direction. So

$$\bar{r}_i \times \bar{F}_{ij} + \bar{r}_i\bar{F}_{ji} = \bar{0}. \qquad \ldots(4)$$

Now the equation of motion of m_i is

$$m_i\ddot{\bar{r}}_i = \bar{F}_{i1} + \bar{F}_{i2} + \cdots = \sum_j \bar{F}_{ij}$$

Multiplying this vectorially by$\bar{r}_i$,

$$\bar{r}_i \times m_i\ddot{\bar{r}}_i = \bar{r}_i \times \sum_j \bar{F}_{ij} = \sum_j \bar{r}_i \times \bar{F}_{ij}$$

Adding all such equations corresponding to $m_1,m_2,m_3 \cdots$,

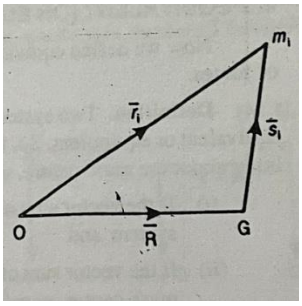
$$\sum_i \bar{r}_i \times m_i \ddot{\bar{r}}_i = \sum_i \sum_j \bar{r}_i \times \bar{F}_{ij}$$

But, in virtue of (4), the terms in the double summation get cancelled in pairs with the exception of the terms which involve the applied forces. So the equation reduces to

$$\sum \bar{r}_i \times m_i \ddot{\bar{r}}_i = \sum \bar{r}_i \times \bar{F}_{ii}. \qquad \ldots(5)$$

Thus, in words,

$$\begin{Bmatrix} \text{The sum of the moments} \\ \text{of the effective forces} \\ \text{about a fixed point } O \end{Bmatrix} = \begin{Bmatrix} \text{The sum of the moments of} \\ \text{the applied forces about } O \end{Bmatrix}. \qquad \ldots(6)$$



Suppose $\bar{s}_i$ is the position vector of m_i with reference to G. Then

$$\bar{r}_i = \bar{R} + \bar{s}_i.$$

Now $(\sum m_i \bar{s}_i)/M$ is the P.V. of the mass centre G with reference to G itself. So it is $\bar{0}$ and hence $\sum m_i \bar{s}_i = \bar{0}$. Now (5) becomes

$$\sum(\bar{R} + \bar{s}_i) \times m_i\left(\ddot{\bar{R}} + \ddot{\bar{s}}_i\right) = \sum(\bar{R} + \bar{s}_i) \times \bar{F}_{ii}.$$

i.e., $(\sum m_i)\bar{R} \times \ddot{\bar{R}} + \bar{R} \times (\sum m_i \ddot{\bar{s}}_i) + (\sum m_i \bar{s}_i) \times \ddot{\bar{R}} + \sum(\bar{s}_i \times m_i \ddot{\bar{s}}_i)$

$$= \bar{R} \times \left(\sum \bar{F}_{ii}\right) + \sum (\bar{s}_i \times \bar{F}_{ii})$$

Since $\sum m_i = M, \sum \bar{F}_{ii} = \bar{F}, M\ddot{\bar{R}} = \bar{F}, \sum m_i \bar{s}_i = \bar{0}, \sum m_i \ddot{\bar{s}}_i = \bar{0}$,

$$\sum \bar{s}_i \times m_i \ddot{\bar{s}}_i = \sum \ddot{\bar{s}}_i \times \bar{F}_{ii}. \qquad \ldots(7)$$

From (7) and (5) it may be noted that the motion of the body relative to the mass centre is the same as it would be if the mass centre is fixed and the same forces act on the body at the same points.

Conditions following equilibrium of a rigid body

If the applied forces keep the rigid body in equilibrium, then the mass centre of the body is at rest and so the velocity $\dot{\bar{R}}$ and acceleration $\ddot{\bar{R}}$ of G are zero and, thus, from (3), we get that

$$\text{The sum of the applied forces} = \bar{0}. \qquad \ldots(8)$$

Also, the body does not have a rotational motion. Thus, from (5) and (7), we get that

$$\left.\begin{array}{l}\text{The vector sum of the moments of the} \\ \text{applied forces about a fixed point or } G\end{array}\right\} = \bar{0}. \qquad \ldots(9)$$

From (8) and (9) it is evident that, if a system of forces keeps a rigid body in equilibrium, then:

(i) The sum of the components of the forces in any direction is zero; and ...(10)

(ii) The sum of the moments of the forces about any line through a fixed point or G is zero. ...(11)

Kinetic energy of a rigid body

Kinetic energy. The kinetic energy of a particle of mass m moving with a velocity v is $1/2\, mv^2$.If the position vector of a particle is $\bar{r}$, then this kinetic energy can be written as $1/2\, m\dot{\bar{r}}.\dot{\bar{r}}$.

Kinetic energy of a rigid body. Let T be the kinetic energy of the rigid body. Then, with the notations used in the previous section, we have:

$$T = 1/2 \sum m_i \dot{\bar{r}}.\dot{\bar{r}} = 1/2 \sum m_i \cdot \left(\dot{\bar{R}} + \dot{\bar{s}}_i\right) \cdot \left(\dot{\bar{R}} + \dot{\bar{s}}_i\right)$$

$$= 1/2 \left[\left(\sum m_i\right) \dot{\bar{R}} \cdot \dot{\bar{R}} + 2\dot{\bar{R}} \cdot \left(\sum m_i \dot{\bar{s}}_i\right) + \sum m_i \dot{\bar{s}}_i \cdot \dot{\bar{s}}_i\right]$$

$$= 1/2\, M \dot{\bar{R}} \cdot \dot{\bar{R}} + 0 + 1/2 \sum m_i \dot{\bar{s}}_i \cdot \dot{\bar{s}}_i$$

$$= \left\{\begin{array}{c}\text{K. E. of a particle of} \\ \text{mass } M \text{ moving with} \\ \text{the mass centre}\end{array}\right\} + \left\{\begin{array}{c}\text{K. E. of the body} \\ \text{due to its motion} \\ \text{relative to the mass centre}\end{array}\right\} \ldots(1)$$

2.3 EQUIVALENT (OR EQUIPOLENT) SYSTEMS OF FORCES

Now we define equivalent systems of forces which are otherwise called equipollent systems of forces.

Definition. Two systems of forces which produce the same motion on a given rigid body are equivalent or equipollent. So, from the equations of the motion of the mass centre andmotion of the body about the mass centre, we get that two systems of forces are equivalent or equipollent

(i) If the vector sum of the forces of one system equals the vector sum of forces of the other systemand

(ii) If the vector sum of the moments of the forces of one system, about any fixed point or the mass centre, equals the vector sum of the moments of the forces of the other system, about the same point on the mass centre.

In symbols, the system of forces $\bar{F}_i$ acting at $\bar{r}_i$ on a rigid body is equivalent to the system of forces$\bar{F}'_j$ acting at $\bar{r}'_j$ on the rigid body if

$$\textstyle\sum_i \bar{F}_i = \sum_j \bar{F}'_j \quad \ldots(1)$$

$$\sum_i \bar{r}_i \times \bar{F}_i = \sum_j \bar{r}'_j \times \bar{F}'_j \quad \ldots(2)$$

Transmissibility of a force. It is evident that two forces equal in magnitude and direction, acting on a rigid body along the same line but at different points, are equivalent. So, a force can be transmitted, without altering its effect on the rigid body, to any other point on its line of action. This is known as the principle of transmissibility of force on a rigid body.

Remark 1. Two forces $\bar{F}_1$ and $\bar{F}_2$ acting at the same point on a rigid body is equivalent to a single force $\bar{F}_1 + \bar{F}_2$ acting at that point because the forces satisfy (1) and (2).

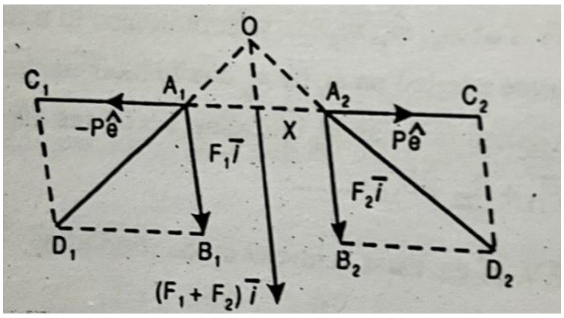
Remark 2. By the principle of transmissibility of a force, we get that, if the lines of action of two forces $\bar{F}_1$ and $\bar{F}_2$ intersect, the forces are equivalent to a single force $\bar{F}_1 + \bar{F}_2$ at the point of intersection.

2.4 PARALLEL FORCES

Computation of the resultant of two parallel forces is not as simple as the computation of the resultant of two intersecting forces.

Parallel forces. Forces whose lines of action are parallel are called parallel forces. If their directions are in the same sense, then they are called like parallel forces; otherwise, they are called unlike parallel forces.

Bookwork 4.2. To find the resultant of two parallel forces acting on a rigid body.



Case (i). Let the forces be like parallel forces, namely $F_1\bar{\iota}$ and $F_2\bar{\iota}$ acting at A_1 and A_2 respectively, where $\bar{\iota}$ is the unit vector in the direction of the forces. Let $\hat{e}$ be the unit vector in the direction of$\overline{A_1A_2}$.Introduce a force $-P\hat{e}$ at A_1 and a force $P\hat{e}$ at A_2. Since these two forces are equal in magnitude and opposite in direction and act along the same line, their introduction will not affect the effects of the given two forces. Let

$$\overline{A_1B_1} = F_1\bar{\iota}, \overline{A_2B_2} = F_2\bar{\iota}, \overline{A_1C_1} = -P\hat{e}, \overline{A_2C_2} = P\hat{e}.$$

Complete the parallelograms $A_1B_1D_1C_1$ and $A_2B_2D_2C_2$.Then the resultant of the two forces $F_1\bar{\iota}$ and $-P\hat{e}$ acting at A_1 is

$$\overline{A_1D_1} = F_1\bar{\iota} - P\hat{e}$$

and the resultant of the forces $F_2\bar{\iota}$ and $P\hat{e}$ acting at A_2 is:

$$\overline{A_2D_2} = F_2\bar{\iota} + P\hat{e}.$$

If the lines A_1D_1 and A_2D_2 intersect at O, then the resultant of these two resultants is

$$\overline{A_1D_1} + \overline{A_2D_2} = (F_1\bar{\iota} - P\hat{e}) + (F_2\bar{\iota} + P\hat{e})$$

$$= (F_1 + F_2)\bar{\iota}$$

acting at O.Note that this resultant is parallel to the original forces.

Point of intersection of the resultant with A_1A_2. From the similar triangles

$$\triangle\, OXA_1, \triangle\, A_1B_1D_1,$$

$$\frac{OX}{XA_1} = \frac{F_1}{P} \ldots (1)$$

Also from the similar triangles $\triangle\, OXA_2$ and $\triangle\, A_2B_2D_2$,

$$\frac{OX}{XA_2} = \frac{F_2}{P} \ldots (2)$$

Dividing (2) by (1), we get

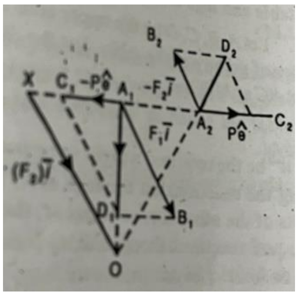
$$\frac{XA_1}{XA_2} = \frac{F_1}{F_2}$$

That is, the line of action of the resultant divides internally A_1A_2 in the ratio $F_2 : F_1$.

Note. In sums this result may be used in the form

$$F_1 \times XA_1 = F_2 \times XA_2.$$

Position vector of X. Let the P.V's of A_1, A_2 be $\bar{r}_1$, $\bar{r}_2$. Since X divides A, A_2 internally in the ratio $F_2 : F_1$, the position vector of X is $\frac{F_1\bar{r}_1+F_2\bar{r}_2}{F_1+F_2}$.



Case (ii). Let the given forces be unlike parallel forces $F_1\bar{\imath}$ and $F_2(-\bar{\imath})$, $(F_1 > F_2)$, acting at A_1 and A_2 respectively. If we adopt the procedure followed in case (i), we see that the steps of case (i) repeat with the only difference that instead of F_2, they have $-F_2$. Thus we get that the resultant of the forces $F_1\bar{\imath}$ and $-F_2\bar{\imath}$ acting at A_1 and A_2 is

$$\{F_1 + (-F_2)\}\bar{\imath}$$

acting at the point which divides A_1A_2 in the ratio $(-F_2) : F_1$, that is, at the point which divides A_1A_2 externally in the ratio $F_2 : F_1$.

Point of application of resultant of many parallel forces

Suppose that a system of parallel forces

$$F_1\bar{\imath}, F_2\bar{\imath}, F_3\bar{\imath}, \dots, F_n\bar{\imath},$$

not necessarily coplanar, act at points whose position vectors are respectively

$$\bar{r}_1, \bar{r}_2, \bar{r}_3, \dots, \bar{r}_n.$$

Then the resultant of $F_1\bar{\imath}$, $F_2\bar{\imath}$ is $(F_1 + F_2)\bar{\imath}$ acting at the point whose P.V. is

$$\frac{F_1\bar{r}_1 + F_2\bar{r}_2}{F_1 + F_2}$$

The resultant of $F_1\bar{\iota}$, $F_2\bar{\iota}$, $F_3\bar{\iota}$ is $(F_1 + F_2 + F_3)\bar{\iota}$ acting at the point whose P.V. is

$$\frac{(F_1 + F_2)\frac{F_1\bar{r}_1 + F_2\bar{r}_2}{F_1 + F_2} + F_3\bar{r}_3}{(F_1 + F_2) + F_3} \text{ or } \frac{F_1\bar{r}_1 + F_2\bar{r}_2 + F_3\bar{r}_3}{F_1 + F_2 + F_3}$$

Similarly, the resultant of the given forces and the P.V. of its point of application are

$$(F_1 + F_2 + \cdots + F_n)\bar{\iota}, \frac{F_1\bar{r}_1 + F_2\bar{r}_2 + \cdots + F_n\bar{r}_n}{F_1 + F_2 + \cdots + F_n}$$

Remark 1. If the forces of earth's gravitation on a system of particles of masses $m_1, m_2, m_3, \ldots$ whose position vectors are $\bar{r}_1, \bar{r}_2, \bar{r}_3, \ldots$ supposed to be parallel, then the position vector of the point, the centre of gravity, through which the resultant gravitation acts, is

$$\frac{m_1 g\bar{r}_1 + m_2 g\bar{r}_2 + \cdots}{m_1 g + m_2 g + \cdots} \text{ or } \frac{m_1\bar{r}_1 + m_2\bar{r}_2 + \cdots}{m_1 + m_2 + \cdots}$$

Centre of gravity, that is, the mass centre is dealt with later.

EXAMPLES

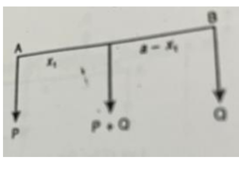
Example 1. Two like parallel forces of magnitudes P, Q, act on a rigid body.If Q is changed to P^2/Q, with the line of action being the same, show that the line of action of the resultant will be the same as it would be, if the forces were simply interchanged.

If the forces, P and $\frac{P^2}{Q}$, act at A, B, then their resultant divides AB internally in the ratio

$$\frac{P^2}{Q} : P \text{or } \frac{P}{Q} = 1 \text{ or } P : Q \ldots (1)$$

For the second case also, the ratio is the same P : Q.Further, all the involved forces and the resultants are parallel to one another.

Example 2. If two like parallel forces of magnitudes P, Q,$(P > Q)$, acting on a rigid body at A, B, are interchanged in position, show that the line of action of the resultant is displaced through a distance $\frac{AB(P-Q)}{P+Q}$.



Let $AB = a$ and let the resultant intersect AB at a distance x_1 from A. Then,

$$x_1 P = (a - x_1)Q$$

$$\therefore x_1 = \frac{aQ}{P+Q} \ldots (1)$$

If the distance in the second case is x_2, then replacing P, Q with Q, P in (1),

$$x_2 = \frac{aQ}{P+Q} \therefore x_2 - x_1 = \frac{a(P-Q)}{P+Q}$$

Example 3. A rod AB of length $a+b$ and weight W has its centre of gravity at a distance a from A. It rests on two parallel knife edges C and D at a distance c apart in the same horizontal plane so that equal portions of the rod project beyond each knife edge. Prove that the reactions of the knife edges on the rod, are respectively

$$\frac{b-a+c}{2c}W, \frac{a-b+c}{2c}W$$

Let the projected lengths be x, x. Then

$$a+b = x+c+x \text{ or } x = {}^1/_2\,(a+b-c).$$

The following forces act on the rod:

(i) Reaction R_1 at C vertically upwards

(ii) Reaction R_2 at D vertically upwards

(iii) Weight W at G vertically downwards.

Since the rod is in equilibrium, the sum of the moments of the forces about any point is zero. Taking moments of these forces about C,

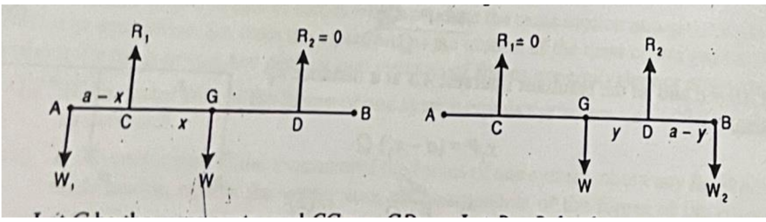
$$-(a-x)W + cR_2 = 0 \text{ or } R_2 = \frac{a-x}{c}W$$

$$\therefore R_2 = \frac{a - {}^1/_2\,(a+b-c)}{c}W = \frac{a-b+c}{2c}W$$

Similarly, taking moments about D, we get R_1 as stated.

Example 4. A uniform plank AB of length $2a$ and weight W is supported horizontally on two horizontal pegs C and D at a distance $d$apart.The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_2 respectively. Show that

$$\frac{W_1}{W+W_1} + \frac{W_2}{W+W_2} = \frac{d}{a}$$



Let G be the mass centre and $CG = x$, $GD = y$.Let R_1, R_2 be the reactions at C,D.When the greatest weight W_1 is placed at A, R_2 is just zero.In this situation taking moments about C,

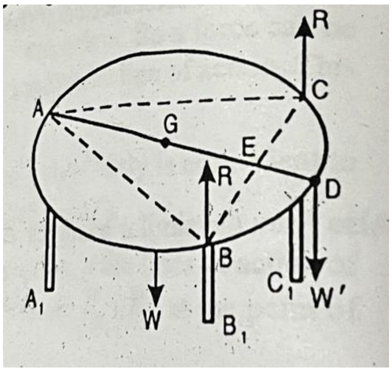
$$(a-x)W_1 - xW = 0 \text{ or} x = \frac{aW_1}{W+W_1}$$

Similarly, in the other case, taking moments about D,

$$YW - (a-y)W_2 = 0 \text{ or } \therefore y = \frac{aW_2}{W+W_2}$$

But $x + y = d$. So the result as stated.

Example 5.A round table of weight W stands on three legs whose upper ends are attached to its rim, so as to form an equilateral triangle. Show that a body whose weight does not exceed W may be placed anywhere on the table without the risk of tilting it.



Let AA_1, BB_1, CC_1 be the legs. Let the diameter AD meet BC at E. The place where the least weight should be placed to topple the table, is D. Suppose that, when a weight W' is placed at D, the table is at the point of tilting. Under this situation the forces acting on the table are

(i) The weight W at G vertically downward

(ii) The weight W' at D vertically downward

(iii) The reactions of the floor R, R at B_1, C_1 vertically upward.

Though the table is on the point of tilting, it is in equilibrium. So the sum of moments of the forces about any point is zero. If moments are taken about E,

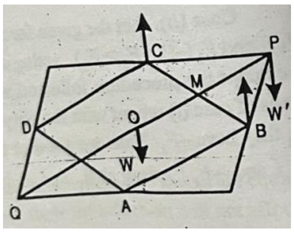
$$\left.\begin{array}{c}\text{Sum of the moments of}\\ \text{the reactions } R, R\end{array}\right\} = 0$$

because $EB = EC$ and, for the moments of W, W',

$$EG \cdot W - ED \cdot W' = 0 \text{ or } W = W'$$

because $EG = ED$ since $AG = r$ and $GE = {}^{r}/_{2}$.

Example 6. A square table stands on four legs placed at the middle points of its sides; find the greatest weight which can be put at one corner of the table without upsetting it, the total weight of the table and legs being W.



Let A, B, C, D be the upper ends of the legs. Let O be the centre of gravity of the table. Let the diagonal QP of the table meet BC at M. Then

$$MB = MC, MO = MP$$

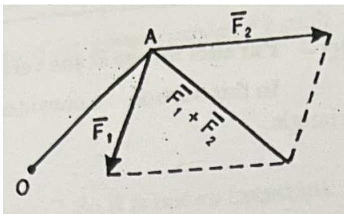
Let W' be the required weight to be put at P. At the moment of tilting the reactions on the legs at A, D are zero. Taking moments of the other forces about M, the sum of the moments of the equal reactions through B, C is zero and

$$MO.W - PM \cdot W' = 0 \text{ or} W' = W.$$

2.5 Varignon's theorem

In this section we bring in the relationship between the sum of the moments of any two coplanar forces and the moment of their resultant.

Bookwork 4.3. (Varignon's theorem). The sum of the moments of two intersecting or parallel forces about any point is equal to the moment of the resultant of the forces about the same point.



Case (i). Intersecting forces. Let the lines of action of the forces $\bar{F}_1$ and $\bar{F}_2$ intersect at A. Then the moments of $\bar{F}_1$ and $\bar{F}_2$ about any point O are

$$\overline{OA} \times \bar{F}_1, \overline{OA} \times \bar{F}_2$$

and their sum is

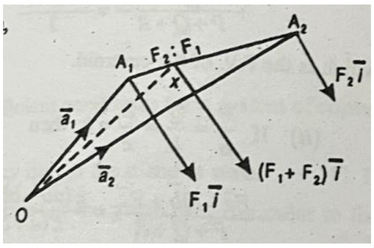
$$\overline{OA} \times \bar{F}_1 + \overline{OA} \times \bar{F}_2.$$

But the resultant of $\bar{F}_1$ and $\bar{F}_2$ is $\bar{F}_1 + \bar{F}_2$ acting at A. So its moment about O is

$$\overline{OA} \times (\bar{F}_1 + \bar{F}_2).$$

Since $\overline{OA} \times \bar{F}_1 + \overline{OA} \times \bar{F}_2 = \overline{OA} \times (\bar{F}_1 + \bar{F}_2)$, the theorem follows for the intersecting forces.

Case (ii). Parallel forces.



Let the parallel forces be $\bar{F}_1 = F_1\bar{\imath}$ and $\bar{F}_2 = F_2\bar{\imath}$ acting at A_1 and A_2. Let $\bar{a}_1, \bar{a}_2$ be the P.V.'s of A_1, A_2 with respect to O. Then, the moments of $\bar{F}_1, \bar{F}_2$ about O are

$$\bar{a}_1 \times F_1\bar{\imath}, \bar{a}_2 \times F_2\bar{\imath}$$

Their sum is

$$\bar{a}_1 \times F_1\bar{\imath} + \bar{a}_2 \times F_2\bar{\imath} = (F_1\bar{a}_1 + F_2\bar{a}_2) \times \bar{\imath}. \qquad \ldots(1)$$

But the resultant of $F_1\bar{\imath}$ and $F_2\bar{\imath}$ is $(F_1 + F_2)\bar{\imath}$ acting at X, where X divides A_1A_2 internally in the ratio $F_2 \cdot F_1$. So the P.V. of X is

$$\frac{F_1\bar{a}_1 + F_2\bar{a}_2}{F_1 + F_2}$$

So the moment of the resultant about O is

$$\overline{OX} \times (F_1 + F_2)\bar{\imath} = \frac{F_1\bar{a}_1 + F_2\bar{a}_2}{F_1 + F_2} \times (F_1 + F_2)\bar{\imath}$$

$$= (F_1\bar{a}_1 + F_2\bar{a}_2) \times \bar{\imath} \ldots (2)$$

From (1) and (2) we get the theorem for parallel forces.

Remark. Varignon's theorem easily extends to any number of coplanar forces. Let $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_n$ be the given forces. Then

$$\text{Mt. of}(\bar{F}_1 + \bar{F}_2) = \text{Mt. of}\bar{F}_1 + \text{Mt. of}\bar{F}_2$$

$$\text{Mt. of}\{(\bar{F}_1 + \bar{F}_2) + \bar{F}_3\} = \text{Mt. of}(\bar{F}_1 + \bar{F}_2) + \text{Mt. of}\bar{F}_3$$

$$= (\text{Mt. of}\bar{F}_1 + \text{Mt. of}\bar{F}_2) + \text{Mt. of}\bar{F}_3$$

$$= \text{Mt. of}\bar{F}_1 + \text{Mt. of}\bar{F}_2 + \text{Mt. of}\bar{F}_3, \text{etc.}$$

2.6 Parallel forces at the vertices of a triangle

In this section we consider the resultant of three like parallel forces acting at the vertices of a triangle.

EXAMPLES

Example 1. Three like parallel forces P, Q, R act at the vertices of a triangle ABC. Show that their resultant passes through

(i) the centroid if $P = Q = R$,

(ii) the incentre if $P/a = Q/b = R/c$.

Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the P.V.'s of A, B, C. Then the resultant passes through the point whose P.V. is

$$\frac{P\bar{a} + Q\bar{b} + R\bar{c}}{P + Q + R}$$

(i) If $P = Q = R$, then

$$\frac{P\bar{a} + Q\bar{b} + R\bar{c}}{P + Q + R} = \frac{\bar{a} + \bar{b} + \bar{c}}{3}$$

which is the P.V. of the centroid.

(ii) If $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} = k$, then

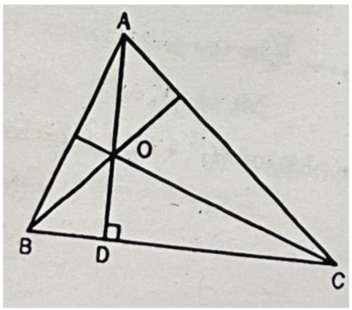
$$\frac{P\bar{a} + Q\bar{b} + R\bar{c}}{P + Q + R} = \frac{k(a\bar{a} + b\bar{b} + c\bar{c})}{k(a + b + c)}$$

$$= \frac{a\bar{a} + b\bar{b} + c\bar{c}}{a + b + c}$$

which is the P.V. of the incentre.

Example 2. Three like parallel forces P, Q, R act at the vertices of a triangle ABC. If their resultant passes through the orthocentre O, show that

$$\frac{P}{\tan A} = \frac{Q}{\tan B} = \frac{R}{\tan C}$$



Let AD be the altitude through A. Now P acts at A. So the resultant of Q, R should act at D such that

$$\frac{BC}{DC} = \frac{R}{Q} \ldots (1)$$

But, from $\triangle ABD$, $\triangle ACD$,

$$BD = \frac{AD}{\tan B}, CD = \frac{AD}{\tan C}$$

Substituting these values in (1), we get

$$\frac{Q}{\tan B} = \frac{R}{\tan C}\text{ , etc.}$$

Example 3. Three like parallel forces P, Q, R act at the vertices of a triangle ABC. If

$$\frac{P}{\tan A} = \frac{Q}{\tan B} = \frac{R}{\tan C} = k$$

show that the resultant of the forces pass through the orthocentre.

Q, R act at B, C. Let their resultant act at X on BC. Then $\overline{AX}$ is given by

$$\overline{AX} = \frac{Q\overline{AB} + R\overline{AC}}{Q + R} = \frac{\tan B\,\overline{AB} + \tan C\,\overline{AC}}{\tan B + \tan C}$$

$$\overline{AX} \cdot \widehat{BC} = \frac{-\tan B \cdot AB \cdot \cos B + \tan C \cdot AC \cdot \cos C}{\tan B + \tan C}$$

$$= \frac{-c \sin B + b \sin C}{\tan B + \tan C} = 0 \text{since } \frac{b}{\sin B} = \frac{c}{\sin C}$$

So AX is perpendicular to BC. Thus AX is the altitude through A. Now P acts at A and $Q + R$ acts at X. So the resultant acts at a point on the altitude. Similarly the resultant acts at points on the other altitudes also. This implies that the resultant acts at the orthocentre.

2.7 FORCES ALONG THE SIDES OF A TRIANGLE

In this section we consider the resultant of forces acting on a rigid body, the forces being along the sides of a triangle.

EXAMPLES

Example 1.P, Q, R are forces along the sides BC, CA, AB of a triangle ABC taken in order. Show that, if their resultant

(i) passes through the incentre, then

$$P + Q + R = 0$$

(ii) passes through the centroid, then

$$\frac{P}{a} + \frac{Q}{b} + \frac{R}{c} = 0 \text{ or} \frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}.$$

(iii) passes through the circumcentre, then

$$P \cos A + Q \cos B + R \cos C = 0.$$

(iv) passes through the orthocentre, then

$$\frac{P}{\cos A} + \frac{Q}{\cos B} + \frac{R}{\cos C} = 0.$$

(i) Incentre I. Let ID, IE, IF be the perpendiculars to the sides. Then

$$ID = IE = IF = r.$$

The resultant passes through the incentre and so its moment about I is zero. Hence the sum of the moments of the given forces about I is zero.

$$\therefore rP + rQ + rR = 0 \;\; or \;\; P + Q + R = 0.$$

Note: If P, Q, R are positive, then

$$P + Q + R \neq 0$$

and so the resultant cannot pass through the incentre. Therefore one or two of P, Q, R must be negative.

(ii) Centroid G. Let AD be a median. Let p_1 be the length of the perpendicular from G to BC. Then the area of $\Delta\ ABC$ is

$$\Delta = \frac{1}{2} BC \times (3p_1)$$

$$\therefore\ p_1 = \frac{2\ \Delta}{3a}, \text{etc.}$$

Now the sum of the moments about G is 0. Therefore

$$P\frac{2\ \Delta}{3a} + Q\frac{2\ \Delta}{3b} + R\frac{2\ \Delta}{3c} = 0\ \ or\ \ \frac{P}{a} + \frac{Q}{b} + \frac{R}{c} = 0.$$

(iii) **Circumcentre S.** Now A is a point on the circumcircle and so BC subtends the angle $2A$ at the centre S.If SD is the perpendicular to BC, finding the area ofΔ BSC, in two different ways,

$$\frac{1}{2} \cdot a \cdot SD = \frac{1}{2} R \cdot R \cdot \sin 2A\ or SD = \frac{R^2 \sin 2A}{a}$$

$$\therefore SD = R^2 \cdot 2 \cdot \frac{\sin A}{a} \cdot \cos A = k \cos A, \text{etc.}$$

Thus, taking moments of the forces about S,

$$SD \cdot P + SE \cdot Q + SF \cdot R = 0\ \ or\ \ k(P\cos A + Q\cos B + R\cos C) = 0$$

$$\therefore P\cos A + Q\cos B + R\cos C = 0.$$

(iv) Orthocentre O. Let AD, BE be two altitudes. Then

$$\angle CBE = 90° - C.$$

So, from $\Delta\ ABD$,

$$\frac{OD}{BD} = \tan(90° - C) = \cot C \qquad \ldots\ldots(1)$$

But, from $\Delta\ ABD$,

$$\frac{BD}{AB} = \cos B. \qquad \ldots.(2)$$

Thus, on multiplying (1) and (2), we get

$$\frac{OD}{AB} = \cot C \cos B\ or OD = \frac{c}{\sin C} \cos C \cos B$$

$$\therefore \qquad OD = k \cos B \cos C, \quad etc.$$

Therefore, taking moments about O,

$$k(P\cos B\cos C + Q\cos C\cos A + R\cos A\cos B) = 0.$$

$$\therefore \qquad \frac{\boldsymbol{P}}{\boldsymbol{\cos A}} + \frac{\boldsymbol{Q}}{\boldsymbol{\cos B}} + \frac{\boldsymbol{R}}{\boldsymbol{\cos C}} = \mathbf{0}.$$

Note. If P, Q, R are positive, one of A, B, C must be obtuse so that its cosine is negative.

Example 2: Three forces P, Q, R act along the sides BC, CA, AB of a triangle ABC. If their resultant passes through the incentre and centroid, then show that

$$\frac{\boldsymbol{P}}{\boldsymbol{a(b-c)}} = \frac{\boldsymbol{Q}}{\boldsymbol{b(c-a)}} = \frac{\boldsymbol{R}}{\boldsymbol{c(a-b)}}.$$

Since the resultant passes through the incentre and centroid, we have respectively

$$P + Q + R = 0 \qquad \ldots.(1)$$

$$\frac{P}{a} + \frac{Q}{b} + \frac{R}{c} = 0 \qquad \ldots.(2)$$

(See the previous example.) Solving (1) and (2) for P, Q, R by the method of cross multiplication, we get

$$\frac{P}{\begin{vmatrix} 1 & 1 \\ \frac{1}{b} & \frac{1}{c} \end{vmatrix}} = \frac{Q}{\begin{vmatrix} 1 & 1 \\ \frac{1}{c} & \frac{1}{a} \end{vmatrix}} = \frac{R}{\begin{vmatrix} 1 & 1 \\ \frac{1}{a} & \frac{1}{b} \end{vmatrix}}$$

$$\frac{P}{1/c - 1/b} = \frac{Q}{1/a - 1/c} = \frac{R}{1/b - 1/a}.$$

Multiplying the denominators by abc, we get the result.

Example 3. Forces $P\,\widehat{BC}, Q\,\widehat{CA}, R\,\widehat{AB}$ act respectively at B, C, A of an equilateral triangle ABC. If their resultant is a force parallel to BC and through the centroid G of the triangle, show that

$$-P = 2Q = 2R.$$

The forces are $P\,\widehat{BC}, Q\,\widehat{CA}, R\,\widehat{AB}$. Their resultant is

$$P\,\widehat{BC} + Q\,\widehat{CA} + R\,\widehat{AB}$$

Since this resultant is parallel to $\widehat{BC}$,

$$\widehat{BC} \times \left(P\widehat{BC} + Q\widehat{CA} + R\widehat{AB}\right) = \bar{0}$$

$$\text{i.e.,}\, \bar{0} + (Q\sin 120° - R\sin 120°)\hat{n} = \bar{0} \text{or } \; Q = R.$$

Since the resultant passes through G, the sum of the moments of the forces about G is zero. Thus

$$p(P+Q+R)=0 \ \ or \ \ P+Q+R=0,$$

where p is the perpendicular distance of the centroid from the sides. But $Q=R$. This leads to the result

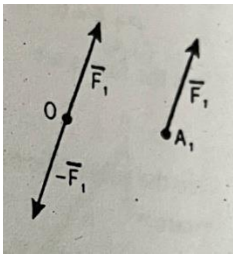
$$-P=2Q=2R.$$

2.8 A SPECIFIC REDUCTION OF FORCES

REDUCTION OF COPLANAR FORCES INTO A FORCE AND A COUPLE

Earlier we saw a system of coplanar forces to reduce to either a single force or a single couple. But in the following bookwork we reduce a system of coplanar forces to a force acting at an arbitrarily chosen point and a couple.

Bookwork 2.1. To show that a system of coplanar forces acting on a rigid body can be reduced to a force at an arbitrarily chosen point and a couple in the plane.



Let $\bar{F}_1, \bar{F}_2, \dots, \bar{F}_n$ be the given forces acting on the rigid body at $A_1, A_2, \dots, A_n$. Let O be an arbitrarily chosen point in the plane. First introduce two forces $\bar{F}_1$ and $-\bar{F}_1$ at this point. This introduction does not alter the effects of the given forces because the resultant of these two forces is a zero force. Now the force $\bar{F}_1$ at A_1 and the force $-\bar{F}_1$ at O from a couple whose moment is $\overline{OA_1} \times \bar{F}_1$. So the force $\bar{F}_1$ at A_1 is equivalent to

(i) A force $\bar{F}_1$ at O and

(ii) A couple with a moment $\overline{OA_1} \times \bar{F}_1$ in the plane.

Similarly the force $\bar{F}_2$ at A_2 is equivalent to a force $\bar{F}_2$ at O from a couple with a moment $\overline{OA_2} \times \bar{F}_2$ and so on. Thus we see that the given force system is equivalent to the system which consists of

(i) n forces. $\bar{F}_1, \bar{F}_2, \dots, \bar{F}_n$ at O and

(ii) n couples with moments $\overline{OA_1} \times \bar{F}_1, \dots, \overline{OA_n} \times \bar{F}_n$ in the plane.

But the resultant of the n forces at O is the single force

$$\bar{F}_1 + \bar{F}_2 + \dots + \bar{F}_n \text{ at } O.$$

The resultant of the n couples is a single couple in the plane with a moment

$$\overline{OA_1} \times \bar{F}_1 + \overline{OA_2} \times \bar{F}_2 + \cdots + \overline{OA_n} \times \bar{F}_n.$$

So the given force system is equivalent to the single force

$$\bar{F}_1 + \bar{F}_2 + \cdots + \bar{F}_n$$

at O and a couple with the moment

$$\overline{OA_1} \times \bar{F}_1 + \overline{OA_2} \times \bar{F}_2 + \cdots + \overline{OA_n} \times \bar{F}_n.$$

But $\overline{OA_1} \times \bar{F}_1, \overline{OA_2} \times \bar{F}_2, \ldots, \overline{OA_n} \times \bar{F}_n$ are the moments of the given forces about O. So the moment of the above couple is the sum of the moments of given forces about O. So, in conclusion, we get that any system of coplanar forces $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_n$ is equivalent to a single force

$$\bar{F}_1 + \bar{F}_2 + \cdots + \bar{F}_n$$

acting at any point in the plane and a couple whose moment is equal to the sum of the moments of $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_n$ about the point.

Invariance of the single force. The single force is the vector sum of the given forces. As such it is independent of O. For brevity we shall denote

$$\bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \cdots + \bar{F}_n \text{by } \bar{F}.$$

Then $\bar{F} \cdot \bar{F}$ or F^2 and the direction of $\bar{F}$ are independent of O.

Invariance of $\mathbf{\bar{F} \cdot \bar{G}}$. The single couple depends on the base point O. So it is not an invariant. We shall denote the moment of this couple by $\bar{G}$. Then

$$\bar{G} = \sum OA_i \times \bar{F}_i.$$

We shall show that $\bar{F}.\bar{G}$ is an invariant. Suppose we reduce the given system to a single force $\bar{F}$at a different point P whose position vector is$\bar{r}$ with reference to O and a couple$\overline{G'}$. Then

$$\overline{G'} = \sum \overline{PA_i} \times \bar{F}_i = \sum (\overline{OA_i} - \overline{OP}) \times \bar{F}_i$$

$$= \sum \overline{OA_i} \times \bar{F}_i - \overline{OP} \times (\sum \bar{F}_i) = \bar{G} - \bar{r} \times \bar{F}.$$

$$\therefore\ \bar{F} \cdot \overline{G'} = \bar{F} \cdot (\bar{G} - \bar{r} \times \bar{F}) = \bar{F} \cdot \bar{G} - \bar{F} \cdot (\bar{r} \times \bar{F})$$

$$= \bar{F} \cdot \bar{G} - [\bar{F}\bar{r}\bar{F}] = \bar{F} \cdot \bar{G} - 0 = \bar{F} \cdot \bar{G}.$$

Thus $\bar{F} \cdot \bar{G}$ is the same for all positions of the point P.

Conditions of equilibrium under coplanar forces

Bookwork 2.2. To show that set of necessary and sufficient conditions for a system of coplanar forces to keep a rigid body in equilibrium are

(i) The sum of the components of the forces along any line in the plane is zero. **C. 1.**

(ii) The sum of the components of the forces along a line in the plane perpendicular to the former is zero. **C. 2.**

(iii) The sum of the moments of the forces about any one point in the plane is zero. **C. 3.**

Necessity part. In this part we have to assume that the system of forces of keep the body in equilibrium and then show that C. 1, C. 2, C. 3 are true.

Let, $\overline{F_1}, \overline{F_2}, \dots, \overline{F_n}$ be the given forces and let O be an arbitrarily chosen point in the plane. Then the given system is equivalent to a force $\overline{F_1} + \overline{F_2} + \cdots + \overline{F_n}$ in the plane at O and a couple in the plane whose moments is the sum of the moments of the forces about O. Now the system keeps the body in equilibrium. So the force at O and the couple must vanish. This leads to the results,

(i) $\overline{F_1} + \overline{F_2} + \cdots + \overline{F_n} = \overline{0}$ ...(1)

(ii) Sum of the moments of the forces about $O = \overline{0}$. ...(2)

Choose arbitrarily a line in the plane through O and let $\bar{\imath}$, be the unit vector along it. Then multiplying (1) scalarly by $\bar{\imath}$.

$$\overline{F_1} \cdot \bar{\imath} + \overline{F_2} \cdot \bar{\imath} + \cdots + \overline{F_n} \cdot \bar{\imath} = 0.$$

This proves C. 1. Similarly, taking the unit vector $\bar{\jmath}$ in the plane, perpendicular to $\bar{\imath}$, and multiplying (1) scalarly by $\bar{\jmath}$, we see that C. 2 is true. Furthermore, we get, from (2), that C. 3 is true.

Sufficiency part. In this part we have to assume that C.1, C. 2, C.3 are true and then showthat the forces keep the body in equilibrium.

As before reduce the system to a single force at O and a single couple. Now, by C.1,

$$\overline{F_1} \cdot \bar{\imath} + \overline{F_2} \cdot \bar{\imath} + \cdots + \overline{F_n} \cdot \bar{\imath} = 0. \quad \therefore \ (\overline{F_1} + \overline{F_2} + \cdots + \overline{F_n}) \cdot \bar{\imath} = 0.$$

Similarly, by C.2,

$$(\overline{F_1} + \overline{F_2} + \cdots + \overline{F_n}) \cdot \bar{\jmath} = 0.$$

Now $\overline{F_1} + \overline{F_2} + \cdots + \overline{F_n}$ cannot be perpendicular to both $\bar{\imath}$ and $\bar{\jmath}$. So the only possibility for the above dot products to vanish, is that $\overline{F_1} + \overline{F_2} + \cdots + \overline{F_n} = \overline{0}$, that is, the single force at O

vanishes. The moment of the couple is the sum of the moments of the given forces about any point. So, by C.3, the couple also vanishes. Hence the body is in equilibrium.

Two more sets of sufficient conditions. The following two bookworks provide two other sets of sufficient conditions for a system of coplanar forces to keep a body in equilibrium.

Bookwork. 2. 3. A set of sufficient conditions for a system of coplanar forces to keep a rigid body in equilibrium is that the sums of the moments of the forces about any three noncollinear points in the plane are zero.

Let the given system consist of forces $\overline{F_1}, \overline{F_2}, \dots, \overline{F_n}$ at $A_1, A_2, \dots, A_n$. Let the sum of the moments of $\overline{F_1}, \overline{F_2}, \dots, \overline{F_n}$ about each of the noncollinear points A, B, C, be zero. Then

$$\sum \overline{AA_\iota} \times \overline{F_\iota} = \bar{0}, \sum \overline{BA_\iota} \times \overline{F_\iota} = \bar{0}, \sum \overline{CA_\iota} \times \overline{F_\iota} = \bar{0}. \qquad \dots(1)$$

Reduce the given system to a force acting at A and a couple. Then the force at A is

$$\overline{F_1} + \overline{F_2} + \dots + \overline{F_n}, \text{ say } \bar{F}$$

and the couple has the moment

$$\sum \overline{AA_\iota} \times \overline{F_\iota}.$$

But, by (1), this moment is zero. So the couple vanishes and the given system reduces to the single force$\bar{F}$ acting at A, that is, the resultant of the given n forces is the single force $\bar{F}$acting at A. But we know that the sum of the moments of a system of forces about any point equals the moment of their resultant about the same point. So the sum of the moments of the given forces $\overline{F_1}, \overline{F_2}, \dots, \overline{F_n}$, about B equals the moment of $\bar{F}$ about B. That is,

$$\overline{BA_1} \times \overline{F_1} + \overline{BA_2} \times \overline{F_2} + \dots + \overline{BA} \times \bar{F}.$$

In this, the left-hand side expression is $\bar{0}$ by (1). So

$$\overline{BA} \times \bar{F} = \bar{0}.$$

Similarly, considering the moments about C, we get

$$\overline{CA} \times \bar{F} = \bar{0}.$$

These two equations imply that either $\bar{F} = \bar{0}$ or $\bar{F}$ is parallel to both $\overline{BA}$, $\overline{CA}$. The latter case is not possible because A, B, C are noncollinear. So $\bar{F} = \bar{0}$. Hence the body is in equilibrium.

Bookwork 2.4. A set of sufficient conditions for a system of coplanar forces to keep a rigid body in equilibrium is that the sums of moments of the forces about any two points A and B are individually zero and that the sum of the components of the forces in any direction not perpendicular to AB is zero.

The first two conditions are the same as the first two conditions of the previous bookwork. So we get that the given system reduces to a single force$\bar{F}$ at A and $\overline{BA} \times \bar{F} = \bar{0}$. The third condition is$\bar{F_1} \cdot \hat{e} + \bar{F_2} \cdot \hat{e} + \cdots + \bar{F_n} \cdot \hat{e} = 0$ or $\bar{F} \cdot \hat{e} = 0$ where $\hat{e}$ is a unit vector not perpendicular to AB. Now considering $\overline{BA} \times \bar{F} = \bar{0}$ and $\bar{F} \cdot \hat{e} = 0$ together, we see that either$\bar{F} = \bar{0}$ or $\bar{F}$ is parallel to $\overline{AB}$ and considering perpendicular to $\hat{e}$. But $\hat{e}$ is not perpendicular to AB. So $\bar{F} = 0$. Hence the body is in equilibrium.

EXAMPLES

Example 1. Prove that, if four forces acting along the sides of a square are in equilibrium, they must be equal in magnitude.

Let the square be $ABCD$ and the unit vectors in the directions of AB, AD be $\bar{\imath}, \bar{\jmath}$. Let the forces and their points of action be

$$P\bar{\imath}, Q\bar{\jmath}, R\bar{\imath}, S\bar{\jmath}\ ;$$

$$\text{A, B, C, D}$$

and let $P > 0$. Due to equilibrium the sum of the forces is $\bar{0}$.

$\therefore\ P\bar{\imath} + Q\bar{\jmath} + R\bar{\imath} + S\bar{\jmath} = \bar{0}$ or $(P+R)\bar{\imath} + (Q+S)\bar{\jmath} = \bar{0}$.

$\therefore\ P + R = 0, Q + S = 0$ or $R = -P, S = -Q$.

Thus the forces are

$$P\bar{\imath}, Q\bar{\jmath}, -P\bar{\imath}, -Q\bar{\jmath}$$

Also the sum of the moments of the forces about any point, say A, is $\bar{0}$. Thus, if a is the side of the square,

$$\bar{O} + \overline{AB} \times (Q\bar{\jmath}) + \overline{AD} \times (-P\bar{\imath}) + \bar{O} = \bar{0}$$

$$i.e., a\bar{\imath} \times (Q\bar{\jmath}) + a\bar{\jmath} \times (-P\bar{\imath}) = \bar{0}$$

$$i.e., aQ\bar{k} + aP\bar{k} = \bar{0} \text{ or } Q = -P.$$

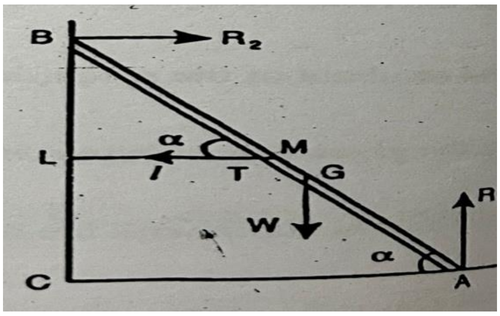
Thus the forces are

$$P\bar{\imath}, -P\bar{\jmath}, -P\bar{\imath}, P\bar{\jmath}.$$

It is evident that their magnitudes are equal.

Example 2. A ladder of length $2a$ and weight W, with its centre of gravity three-eighths of the way up, stands on a smooth horizontal plane resting against a smooth vertical wall and the midpoint is tied to a point in the wall by a horizontal rope of length l. Find the tension of the rope.

The figure is self explanatory. The forces on the ladder are



(i) Reaction R_1 at A

(ii) Reaction R_2 at B

(iii)Weight Wat G

(iv)Tension T at M along ML.

Considering the horizontal components, we get

$$T = R_2.$$

We shall not consider the vertical components. Let α be the angle made by the ladder with the horizontal. Now the perpendicular distances of the forces W,T,R_2 from A are

$$AG\cos\alpha\,, AM\sin\alpha\,, AB\sin\alpha$$

$$2a\cdot {}^{3}/_{8}\cdot\cos\alpha\,, a\sin\alpha\,, 2a\sin\alpha.$$

Taking moments of all the four forces about A

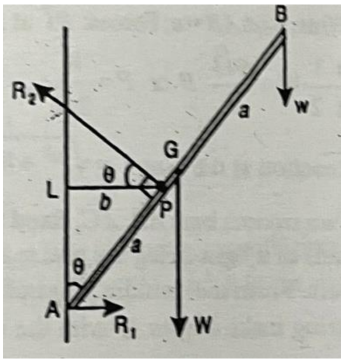
$$\left(2a\cdot {}^{3}/_{8}\cdot\cos\alpha\right)W + (a\sin\alpha)T = (2a\sin\alpha)R_2$$

$$i.e.,{}^{3}/_{4}\cot\alpha\, W + T = 2R_2 \text{ or } {}^{3}/_{4}\cot\alpha\, W + T = 2T$$

$$i.e., T = \frac{3}{4}\cot\alpha\, W = \frac{3}{4}\frac{l}{\sqrt{a^2-l^2}}W \text{ from } \triangle\, BLM.$$

Example 3. A uniform rod of length $2a$ and weight W is resting on a fixed rail parallel to a smooth vertical wall with one of its ends pressing the wall and to its other end is attached a weight$w$. The distance of the rail from the wall is b. If the rod is in equlibrium in a vertical plane, show that the angle θ made by the rod with the vertical is given by

$$\sin\theta = \left[\frac{bW + w}{aW + 2w}\right]^{1/3}$$



The figure is selfexplanatory. We can avoid R_1 by resolving the forces vertically and taking moments about A. Thus, resolving vertically,

$$R_2 \sin\theta = W + w. \qquad \ldots(1)$$

Taking moments about A,

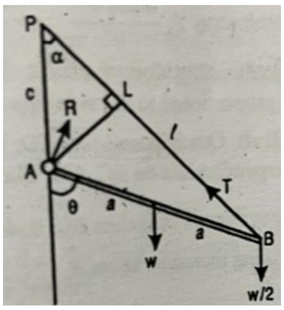
$$AP \cdot R_2 = AG \sin\theta \cdot W + AB \sin\theta \cdot w, AP = \frac{LP}{\sin\theta} = \frac{b}{\sin\theta}$$

$$\therefore \frac{b}{\sin\theta} R_2 = a \sin\theta\, W + 2a \sin\theta\, w$$

$$\therefore \frac{b}{\sin\theta} \frac{W+w}{\sin\theta} = a \sin\theta\, W + 2a \sin\theta\, w \text{ by (1).}$$

This gives $\sin\theta$ as in the result.

Example 4. A uniform rod of weight w is movable in a vertical plane about a hinge at one end, and at the other end is fastened weight$^{w}/_{2}$. The latter end is fastened by a string of length l to a point at a height c vertically over the hinge. Show that the tension of the string islw/c.



Let AB be the rod, $AB = 2a$ and BP, the string. Let the rod and the string make anglesθ, αwith the vertical. Four forces act on the rod. They are

(i) Weights $w \cdot {}^{w}/_{2}$

(ii) Tension$T$

(iii)Reaction R at$A$.

Let AL be the perpendicular from A to the string. Taking moments about$A$, thus avoiding R,

$$-a \sin \theta \, w - 2a \sin \theta \, (w/2) + T \cdot AL = 0.$$

$$\therefore T = \frac{2a \sin \theta}{AL} . w. \qquad \ldots(1)$$

From the right angled $\triangle\, APL$,

$$AL = c \sin \alpha. \qquad \ldots(2)$$

Also, using the sine formula for $\triangle\, ABP$,

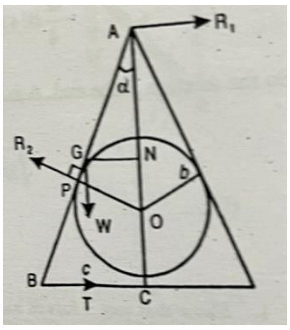
$$\frac{l}{\sin(180° - \theta)} = \frac{2a}{\sin \alpha} \text{ or } 2a \sin \theta = l \sin \alpha. \qquad \ldots(3)$$

When (2) and (3) are used, (1) becomes

$$T = \frac{(l \sin \alpha) w}{(c \sin \alpha)} \text{ or } T = \frac{lw}{c}$$

Example 5. Two equal uniform rods, each of weight W and length a are freely jointed at one extremity, and rest symmetrically in a vertical plane in contact with a smooth horizontal cylinder of radius b, the axis of the cylinder being at right angles to the plane of the rods. Their other extremities are connected by an inextensible string of length $2C$. Show that the tension of the string is

$$W \left(\frac{ab}{c^2} - \frac{c}{2\sqrt{a^2 - c^2}} \right).$$



Let AB be one rod touching the cylinder at P. Let O be the centre of the cylinder. Let G be the midpoint of AB and GN, the perpendicular from G to the vertical through A. The forces on AB are

(i) Reaction R_1, at A

(ii) Reaction R_2, at P

(iii) Weight W at G

(iv) Tension T at B

Let α be the angle BAC. Now the perpendicular distances of the forces R_2, W, T from A are

$$AP, GN, AC$$

$$i.e., OP \cot \alpha \,, AG \sin \alpha \,, AB \cos \alpha$$

$$i.e., b\cot\alpha\,, {}^{1}/_{2}\, a\sin\alpha\,, a\cos\alpha.$$

So, taking moments of the forces about A,

$$-b\cot\alpha\, R_2 + {}^{1}/_{2}\, a\sin\alpha\,, W + a\cos\alpha\, T = 0$$

$$T = \frac{b\cot\alpha\, R_2 - {}^{1}/_{2}\, a\sin\alpha\, W}{a\cos\alpha} = \frac{b}{a}\operatorname{cosec}\alpha\, R_2 - {}^{1}/_{2}\tan\alpha\, W$$

Due to symmetry the reaction on the rod at A is horizontal. Resolving the forces vertically, thus avoiding R_1,

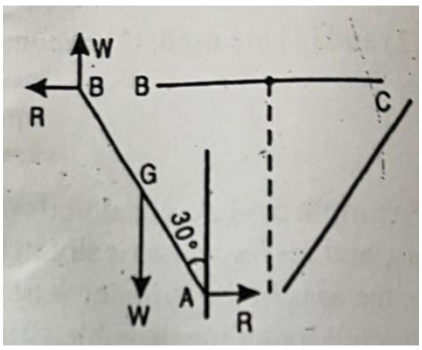
$$R_2 \sin\alpha = W \text{ or } R_2 = \frac{W}{\sin\alpha}$$

$$\therefore T = \frac{b}{a} cosec^2\alpha\, W - \frac{1}{2}\tan\alpha\, W = \left(\frac{b}{a}\frac{a^2}{c^2} - \frac{1}{2}\frac{c}{\sqrt{a^2-c^2}}\right) W$$

$$= \left(\frac{ab}{c^2} - \frac{1}{2}\frac{c}{\sqrt{a^2-c^2}}\right) W$$

Example 6. Three equal uniform rods, each of weight W, are smoothly jointedsoasto forman equilateral triangle ABC. The system is supported at the middle point of the rod BC. Show that the reaction at the hinge at A is $\frac{W\sqrt{3}}{6}$.

Show also that the action at each of the other hinges is $W\sqrt{\frac{13}{12}}$.



The reactions at the end A on the rods BA, CA should be similar and opposite. So they should be only horizontal. Let them be R, R. We shall consider the forces on the rod ABonly.The weight $W$of the rod acts at G vertically downward. But there are only three forces on the rod AB. Since the rod is in equilibrium the third force, the reaction at B, should have its horizontal and vertical components as R, Wasshown in the figure.

If $BC = a$, then the vertical through G meets BC at a distance${}^{a}/_{4}$ from A.

Taking moments about B,

$$-\frac{a}{4}W + (a\cos 30°)R = 0 \text{ or } R\frac{\sqrt{3}}{2} = \frac{W}{4} \text{ or } R = \frac{W}{2\sqrt{3}}$$

So the reaction at the end $A$is $\frac{W}{2\sqrt{3}}$. The reaction at the end Bandalso at the end $C$is

$$\sqrt{R^2 + W^2}\text{or } \sqrt{\frac{W^2}{12} + W^2}\text{or } \sqrt{\frac{13}{12}}W$$

PROBLEMS INVOLVING FRICTIONAL FORCES

We know that, when a body is simply in equilibrium, that is, not in limiting equilibrium, the friction called into play is less than μR, whereμ is the coefficient of friction and R is the normal reaction. But, if the equilibrium is limiting, then

$$F = \mu R, \tan\lambda = \mu,$$

Where

(i) $F = \mu R$ is the frictional force

(ii) R is the normal reaction

(iii)μ is the coefficient of friction

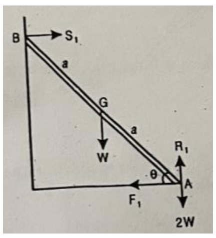
(iv)$\lambda$is the angle of friction.

The angle of friction is the angle made by the reaction with the normal.

In problems involving limiting friction the method of solving them is two-fold; one is to keep the reaction as a single force as such and the other is to resolve it into its components, namely, the normal reaction and the limiting friction.

EXAMPLES

Example 1. A uniform ladder AB rests against a smooth wall at B and upon a rough ground at A. Boy whose weight is twice that of the ladder, climbs it. Prove that, the force of friction when he is at the top of the ladder, is five times as great as when he is at the bottom.



The ladder is not in limiting equilibrium. So, in the first case, the forces at A are

(i) Friction F_1 horizontally

(ii) Normal reaction R_1 vertically

(iii) Weight of the boy $2W$

As in the figure. The vertical components of the forces given

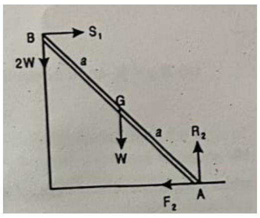
$$R_1 = 3W. \quad \ldots(1)$$

If θ is the angle of inclination of the ladder to the horizontal and $AB = 2a$, taking moments about B,

$$a\cos\theta\, W + 4a\cos\theta\, W + 2a\sin\theta\, F_1 - 2a\cos\theta\, R_1 = 0.$$

$$5W + 2\tan\theta\, F_1 = 2R_1 \text{ or } 5W + 2\tan\theta\, F_1 = 6W \text{ by (1)}$$

$$F_1 = \frac{W}{2\tan\theta} \quad \ldots(2)$$



In the second case, the forces at A are

(i) Friction F_2 horizontally,

(ii) Normal reaction R_2 vertically.

Now the vertical components of the forces gives.

$$R_2 = 3W. \quad \ldots(3)$$

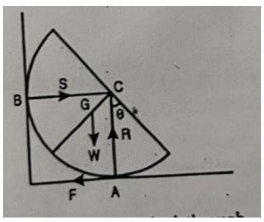
Taking moments about B, and dividing by $a\cos\theta$,

$$W + 2\tan\theta\, F_2 = 2R_2 \text{ or } W + 2\tan\theta\, F_2 = 6W \text{ by (3)}$$

$$F_2 = \frac{5W}{2\tan\theta} \ldots(4)$$

From (2) and (4), we get the required result.

Example 2. A solid hemisphere rests on a rough horizontal plane and against a smooth vertical wall. Show that, if the coefficient of friction μ is greater than $3/8$, then the hemispheres can rest in any position and if it is less, the least angle that the base of the hemisphere can makewith the vertical is $\cos^{-1}\frac{8\mu}{3}$.



Let, in the equilibrium position, the points of contact with the plane and the wall be A, B. The forces on the body are

(i) Friction F at A

(ii) Reaction R at A

(iii)Reaction S at B

(iv)Weight W at G

as in the figure. Then, from the horizontal and vertical components,

$$S - F = 0, R - W = 0 \text{ or } S = F, R = W.$$

Now $CG = {}^{3a}/_{8}$, where a is the radius of the sphere and the distance of G from the vertical through A is

$$CG \cos\theta \text{ or } {}^{3a}/_{8} \cos\theta,$$

Where θ is the angle made by the base with the vertical. Thus, taking moments about A,

$${}^{3a}/_{8} \cos\theta\, W - aS = 0 \text{ or } {}^{3a}/_{8} \cos\theta\, R = aF.$$

$$\therefore \frac{F}{R} = \frac{3}{8}\cos\theta$$

(i) So the equilibrium is possible for any inclination so long the coefficient of friction is large such that

$$\mu > \tfrac{3}{8} cos\,\theta \text{ or} \tfrac{8\mu}{3} > cos\,\theta. \qquad \ldots(1)$$

But the maximum value for $cos\,\theta$ is 1. Thus (1) becomes

$$\tfrac{8\mu}{3} > 1 \text{ or} \mu > \tfrac{3}{8}.$$

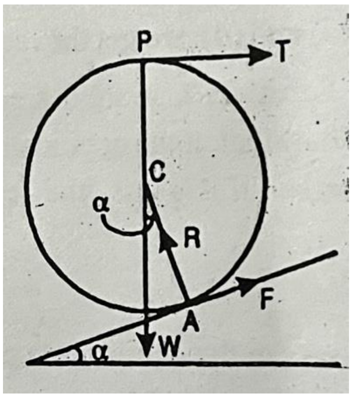
(ii) If $\mu < \frac{3}{8}$ and if the equilibrium is limiting, then $F = \mu R$ and (1) gives

$$\mu = \frac{3}{8} cos\,\theta \text{ or } cos\,\theta = \frac{8\mu}{3}$$

$$\text{i.e., } \theta = \cos^{-1}\tfrac{8\mu}{3}$$

which is the required least angle because, for a still smaller angle, the friction required is larger which cannot be attained.

Example 3. A sphere of weight W resting on a rough inclined plane of inclination α is kept in equilibrium by a horizontal string attached to the highest point of the sphere. Show that the angle of friction is greater than ${}^{\alpha}/_{2}$ and that the tension of the string is $W \tan {}^{\alpha}/_{2}$.



Let A be the point of contact, C the centre and P the topmost point. Forces on the sphere are

(i) Friction F at A up the plane

(ii) Reaction R at A normal to the plane

(iii) Tension T horizontally at P

(iv) Weight W vertically downward.

Taking moments about C,

$$aF - aT = 0$$

$$\therefore F = T. \qquad \ldots(1)$$

Resolving the forces horizontally and vertically,

$$T + F\cos\alpha = R\sin\alpha, \qquad \ldots(2)$$

$$R\cos\alpha + F\sin\alpha = W. \qquad \ldots(3)$$

From (1) and (2),

$$\frac{F}{R} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{2\sin{}^{\alpha}/_{2}\cos{}^{\alpha}/_{2}}{2\cos^2{}^{\alpha}/_{2}} = \tan{}^{\alpha}/_{2}$$

μ should be large enough so that ${}^{F}/_{R} < \mu$. That is,

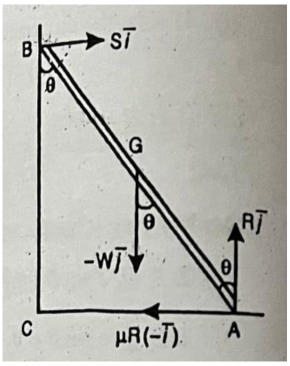
$$\tan{}^{\alpha}/_{2} < \tan\lambda \text{ or } {}^{\alpha}/_{2} < \lambda.$$

Remembering that $F = T$ and solving for T, we get

$$T = \frac{\sin\alpha}{1+\cos\alpha}W = W\tan\frac{\alpha}{2}$$

Example 4. A uniform ladder AB rests in limiting equilibrium with the end A on a rough floor, the coefficient of friction being μ and with the other end B against a smooth vertical wall.

Show that, ifθ is the inclination of the ladder to the vertical then $\tan\theta = 2\mu$. If $\theta = 30°$, find μ.



Method 1.(i) Now Ahasatendencyto move away from the wall. So the frictional force at A is towards the wall. Thus, if $\bar{\imath}, \bar{\jmath}$ are the unit vectors along CA and CB, the forces acting on the ladder are

(i) $-\mu R\bar{\imath}$, $R\bar{\jmath}$ at A

(ii) $S\bar{\imath}$ at B

(iii) $-W\bar{\jmath}$ at G

The sum of the $\bar{\imath}$ components and the sum of $\bar{\jmath}$components are zero. Therefore

$$S - \mu R = 0, R - W = 0.$$

Solving these two equations,

$$R = W, S = \mu W.$$

If $AB = 2a$, taking moments about A of the forces,

$$a\sin\theta\, W - 2a\cos\theta\, S = 0 \text{or} \tan\theta = \frac{2S}{W} = \frac{2\mu W}{W} = 2\mu$$

(iii) When $\theta = 30°$,

$$\tan 30° = 2\mu or \mu = \frac{1}{2\sqrt{3}}$$

Remark. The magnitude of the reaction acting on the rod at A, is

$$\sqrt{R^2 + \mu^2 R^2} = R\sqrt{1 + \mu^2} = W\sqrt{1 + \mu^2}.$$

Method 2. Let the weight and the reaction at B intersect at O. Then the reaction at A passes through O. The reaction is inclined to the normal to the floor, that is, to the vertical at an angle λ, where λ is the angle of friction and $\mu = \tan\lambda$.

In the triangle AOB, $BG : GA = 1 : 1$ and

$$\angle OGB = \theta; \angle GOB = 90°; \angle GOA = \lambda.$$

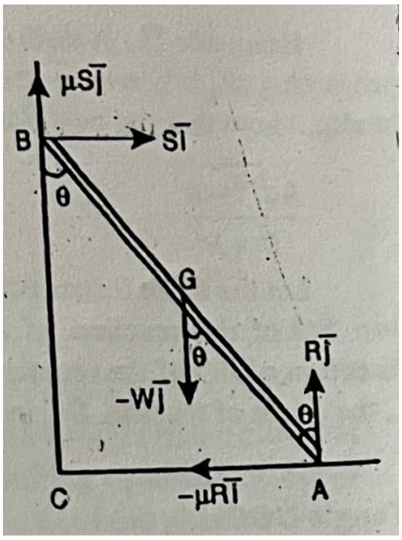
So, by cotangent formula,

$$(1+1)\cot\theta = \cot\lambda - \cot 90°.$$

$$2\cot\theta = \cot\lambda \text{ or } \tan\theta = 2\tan\lambda = 2\mu.$$

Example 5. Find the inclination θ to the vertical of a uniform ladder AB of length $2a$ and weight W which is in limiting equilibrium having contact with a rough horizontal floor and a rough vertical wall, the coefficients of friction being μ.

Show that the greatest inclination of the ladder to the vertical is 2λ.



(i) Now A has a tendency to move away from the wall and Bhas a tendency to move downwards. Let the forces acting on the ladder be

(i) $-\mu R\bar{\imath}$, $R\bar{\jmath}$ at A

(ii) $S\bar{\imath}$, $\mu S\bar{\jmath}$ at B

(iii)$-W\bar{\jmath}$ at G

Since the ladder is in equilibrium, the sums of the components of these forces in the $\bar{\imath}$, $\bar{\jmath}$ directions are zero. So

$$S - \mu R = 0,\ R + \mu S - W = 0.$$

Solving for R and S,

$$R = \frac{W}{1+\mu^2}, S = \frac{\mu W}{1+\mu^2}$$

Further, finding the moments of the forces about A,

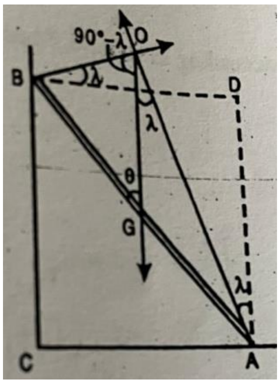
$$a\sin\theta\, W - 2a\sin\theta\,\mu S - 2a\cos\theta\, S = 0$$

$$\text{i.e., } (W - 2\mu S)\sin\theta - (2S)\cos\theta = 0$$

$$\therefore \tan\theta = \frac{2S}{W - 2\mu S} = \frac{2\mu}{1-\mu^2}$$

(ii)Since $\mu = \tan\lambda$, $\tan\theta = \frac{2\mu}{1-\mu^2} = \frac{2\tan\lambda}{1-\tan^2\lambda} = \tan 2\lambda$ or $\theta = 2\lambda$.

Method 2.



Now the forces on the rod are the reactions of the floor and the wall at A and B and the weight. Let them meet at O.Then AO is inclined to the normal AD to the floor at the angle λ and BO is inclined to the normal BD to the wall at the angle λ. So, in the ΔOAB,

$$BG : GA = 1 : 1,$$

$$\angle OGB = \theta, \angle BOG = 90° - \lambda, \angle AOG = \lambda.$$

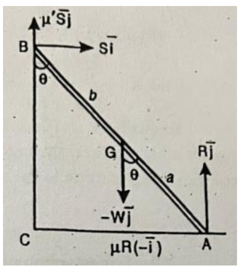
Thus, by the cotangent formula,

$$(1+1)\cot\theta = \cot\lambda - \cot(90° - \lambda) = \cot\lambda - \tan\lambda$$

$$= \frac{1}{\mu} - \mu = \frac{1-\mu^2}{\mu}$$

$$\therefore \tan\theta = \frac{2\mu}{1-\mu^2}$$

Example 6. A ladder which stands on a horizontal ground leaning against a verticalwall, has its centre of gravity at distances a and b from its lower and upper ends respectively.Show that, if the ladder is in limiting equilibrium, and if μ and μ' are the coefficients of friction at the lower and upper contacts, its inclination θ to the vertical is given by $\tan\theta = \frac{(a+b)\mu}{a - b\mu\mu'}$.



The figure is self explanatory. The sums of the $\bar{\imath}$ components and $\bar{\jmath}$ components are zero. So we have

$$S - \mu R = 0,$$
$$R + \mu' S - W = 0.$$

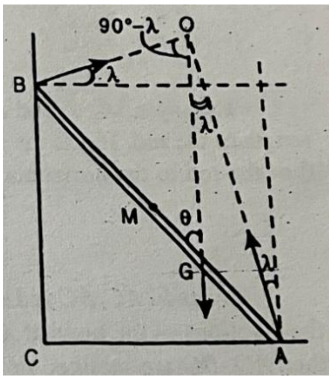
Solving these, we get

$$R = \frac{1}{1 + \mu\mu'} W,$$
$$S = \frac{\mu}{1 + \mu\mu'} W.$$

Further moments taking about A,

$$a \sin\theta\, W - (a + b)\cos\theta\, S - (a + b)\sin\theta\, \mu' S = 0$$
$$\text{or } aW \tan\theta - (a + b)S - (a + b)\mu' S \tan\theta = 0.$$
$$\therefore\ \tan\theta = \frac{(a + b)S}{aW - (a + b)\mu' S} = \frac{(a + b)\mu}{a(1 + \mu\mu') - (a + b)\mu\mu'} = \frac{(a + b)\mu}{a - b\mu\mu'}$$

Example 7. A ladder of length $2a$ is in contact with a wall and a horizontal floor, the angle of friction being λ at each contact. If the centre of gravity of the ladder is at a distance ka below the midpoint, show that in the limiting equilibrium, the inclination θ to the vertical is given by $\cot\theta = \cot 2\lambda - k cosec 2\lambda$.



The forces acting on the ladder are the reactions of the floor and the wall inclined to the normais to them at an angle λ and the weight. If they meet at O, then both AO and BO are inclined to the vertical and horizontal at the same angle λ. So, in the triangle ABO,

$$\angle BGO = \theta, \angle AOG = \lambda, \angle BOG = 90° - \lambda.$$
$$BG : GA = a + MG : a - MG = a + ka : a - ka = 1 + k : 1 - k.$$

So, by cotangent formula,

$$\{(1 + k) + (1 - k)\}\cot\theta = (1 - k)\cot\lambda - (1 + k)\cot(90° - \lambda).$$

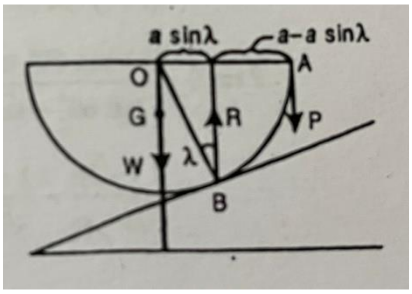
$$\text{or } 2\cot\theta = (1-k)\cot\lambda - (1+k)\tan\lambda$$
$$= (\cot\lambda - \tan\lambda) - k(\cot\lambda + \tan\lambda).$$

So the result follows as stated since it can be shown that

$$\cot\lambda - \tan\lambda = 2\cot 2\lambda, \cot\lambda + \tan\lambda = 2cosec 2\lambda.$$

Example 8. A solid hemisphere of weight W rests in limiting equilibrium with its curved surface on a rough inclined plane, its plane face being kept horizontal by a weight P attached to a point A in its rim. Prove that the coefficient of friction is

$$\mu = \frac{P}{\sqrt{W(2P+W)}}$$



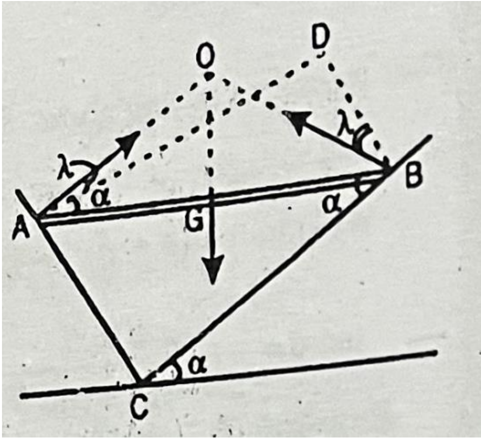
Let O be the centre and a, the radius of the sphere and B, the point of contact of the hemisphere with the inclined plane. The forces acting on the hemisphere are its weight W along OG, the attached weight Pat$A$ and the reaction R of the plane. Since W and $P$are vertical, R also should be vertical. R should be inclined to BO, the normal to the plane at B at an angle equal to the angle frictionλ where$\mu = \tan\lambda$. Now taking moments about B,

$$W(a\sin\lambda) - P(a - a\sin\lambda) = 0 \; or \; \sin\lambda = \frac{P}{W+P}$$

$$\therefore \; \tan\lambda = \frac{P}{\sqrt{(W+P)^2 - P^2}} \; or \; \mu = \frac{P}{\sqrt{W^2 + 2WP}}$$

Example 9. A rod is in limiting equilibrium resting horizontally with its ends on two inclined planes which are at right angles and one of which makes an angle α ($< 45°$) with the horizontal. If the coefficient of friction is the same for both the ends, show that

$$\mu = \frac{1-\tan\alpha}{1+\tan\alpha}$$



The forces acting on the rod are the reactions of the planes and its own weight. Let the reactions meet at O. Then the weight should act through O so that OG is vertical. But AB is horizontal and $AO = OB$. So$\triangle$ AOB is isosceles and

$$\angle OAG = \angle OBG.$$

But$A$has a tendency to slide downwards and$B$has a tendency to slide upwards because $\alpha < 45°$. As such AO and BO are inclined to the normals AD and BD to the plane at the same angle of friction λ as shown in the figure. Now

$$\angle OAG = \alpha + \lambda, \angle OBG = 90° - (\alpha + \lambda).$$

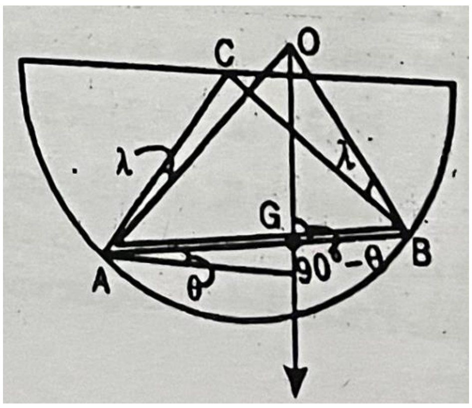
$$\alpha + \lambda = 90° - (\alpha + \lambda) \text{or } \alpha + \lambda = 45°.$$

$$\tan(\alpha + \lambda) = 1 \text{ or} \tan\alpha + \tan\lambda = 1 - \tan\alpha \tan\lambda.$$

$$\tan\alpha + \mu = 1 - \mu \tan\alpha \text{ or}\mu(1 + \tan\alpha) = 1 - \tan\alpha.$$

$$\mu = \frac{1 - \tan\alpha}{1 + \tan\alpha}$$

Example 10. A rod AB rests within a fixed hemispherical bowl whose radius is equal to the length of the rod. If μ is the coefficient of friction, show that, in limiting equilibrium, the inclination θ of the rod to the horizontal is given by $\tan\theta = \frac{4\mu}{3-\mu^2}$.



The radii AC, BC and the rod AB form an equilateral triangle. If the reactions of the bowl at A, B and the weight of the rod meet at O, then AO, BO are inclined to AC, BC at the same angle of friction λ. Now in the triangle AOB, $AG : GB = 1 : 1$ and

$$\angle OGB = 90° - \theta, \angle GAO = 60° - \lambda, \angle GBO = 60° + \lambda.$$

Therefore, by cotangent formula,

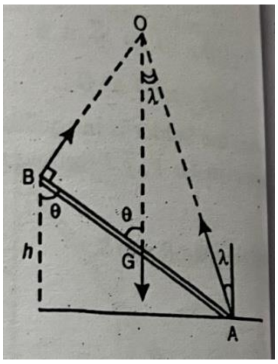
$$(1+1)\cot(90° - \theta) = \cot(60° - \lambda) - \cot(60° + \lambda).$$

$$\therefore 2\tan\theta = \frac{1 + \tan 60° \tan\lambda}{\tan 60° - \tan\lambda} - \frac{1 - \tan 60° \tan\lambda}{\tan 60° + \tan\lambda}$$

$$= \frac{1+\sqrt{3}\mu}{\sqrt{3}-\mu} - \frac{1-\sqrt{3}\mu}{\sqrt{3}+\mu} = \frac{8\mu}{3-\mu^2}$$

$$\therefore \tan\theta = \frac{4\mu}{3-\mu^2}$$

Example 11. A uniform ladder of length l rests on a rough horizontal ground with its upper end projecting slightly over a smooth horizontal rod at a height h above the ground. If the ladder is about to slip, show that the coefficient of friction is equal to $\frac{h\sqrt{l^2-h^2}}{l^2+h^2}$.



Let the three forces acting on the ladder meet at O. Then the direction BO of the reaction of the peg B is perpendicular to AB and the direction of AO of the reaction of the ground is inclined to the vertical at λ, the angle of friction. So, in the triangle ABO,

$$AG : GB = 1 : 1.$$

If angle OGB is θ, then

$$\angle OBA = 90°, \angle OAB = \theta - \lambda.$$

So, by cotangent formula,

$$(1+1)\cot\theta = \cot(\theta - \lambda) - \cot(90°) = \cot(\theta - \lambda).$$

$$\therefore \tan\theta = 2\tan(\theta - \lambda) = 2\frac{\tan\theta - \tan\lambda}{1 + \tan\theta\tan\lambda} = 2\frac{\tan\theta - \mu}{1 + \mu\tan\theta}$$

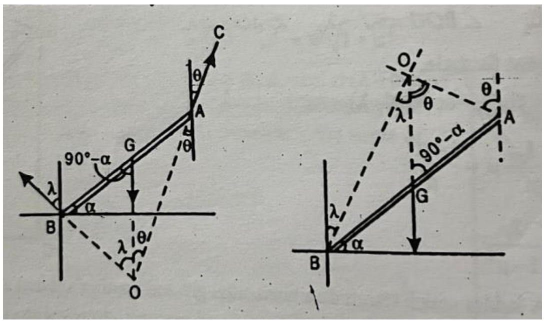
This gives $\mu = \frac{\tan\theta}{2+\tan^2\theta}$

But $\tan\theta = \frac{\sqrt{l^2-h^2}}{h}$.So the result as stated.

Example 12. A rod AB is supported at an inclination a to the horizontal with its lower end B on a rough, horizontal plane by a string AC attached to A. Show that, if μ is the coefficient of friction, then the greatest inclination θ of the string to the vertical is given by

$$\cot\theta = \frac{1}{\mu} + 2\tan\alpha \; or \; \cot\theta = \frac{1}{\mu} - 2\tan\alpha,$$

according as the end Bhas a tendency to move closer to the string or not.



The figures correspond to the first and second cases respectively. Let λ be the angle of friction. Then$\mu = \tan\lambda$. Now the tension acts along AC and the weight along the vertical through G. Let them meet at O. Then the reaction of the plane acts along OB. It should be inclined to the plane at an angle λ. In triangle AOB, $AG : GB = 1 : 1$ and

$$\angle AOG = \theta, \angle BOG = \lambda.$$

Also in the first and second figures

$$\angle OGB = 90° - \alpha, \angle OGA = 90° - \alpha$$

respectively. So, in these cases, by cotangent formula,

$$(1+1)\cot(90° - \alpha) = \cot\theta - \cot\lambda = \cot\theta - \frac{1}{\mu}$$

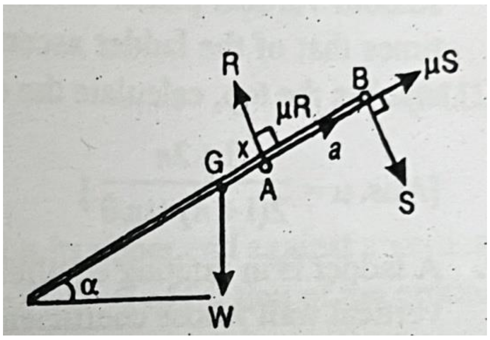
$$(1+1)\cot(90° - \alpha) = \cot\lambda - \cot\theta = \frac{1}{\mu} - \cot\theta$$

which lead to the required results.

Example 13. Two equally rough pegs A, B are at a distance a apart in a straight line inclined at an angle α to the horizontal. A rod passes over A but under B and rests in limiting equilibrium. Show that, if λ is the angle of friction, then the distance$x$ between the mass centre G of the rod and A is $x = {}^{a}/_{2}(\cot\lambda\tan\alpha - 1)$.

Show that the length of the shortest rod which will rest in such a position is

$a(\cot\lambda\tan\alpha + 1)$.



The reactions R, S and the friction $\mu R, \mu S$ are perpendicular and along the rod as in the figure. The components of$W$ in these directions are

$$W \cos \alpha , W \sin \alpha.$$

$$\therefore W \cos \alpha + S = R, W \sin \alpha = \mu R + \mu S$$

$$\therefore S = \frac{W(sin\alpha - \mu cos\alpha)}{2\mu}$$

Taking moments about A,

$$(xcos\alpha)W = aSorx = \frac{aS}{Wcos\alpha}.$$

$$x = \frac{a}{W\, cos\alpha}\frac{W(sin\alpha - \mu cos\alpha)}{2\mu}$$

$$= \frac{a(tan\alpha - \mu)}{2\mu} \qquad by(1)$$

$$= \frac{a(tan\alpha - tan\lambda)}{2\, tan\lambda}$$

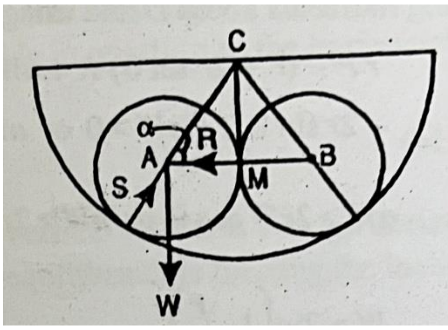
$$= \frac{a}{2}(tan\alpha cot\alpha - 1).$$

The rod is of least length when its upper end is just below B. At that moment its length is

$$2GB \;=\; 2(x + a) = 2\left[\frac{a}{2}(tan\alpha\, cot\lambda - 1) + a\right]$$

$$= a\,(tan\alpha\ cot\lambda\ + 1).$$

Example 14. Two smooth equal spheres of radius$a$and weight$W$ lie within a fixed smooth spherical bowl of radius b. Prove that the pressure between them is$\frac{aW}{\sqrt{b^2-2ab}}$. If$b - 3a$, show that the ratio of reaction between the spheres to the reaction between a sphere And the bowl is$\frac{1}{2}$.



The figure is self-explanatory. In it the force on the left sphere only are marked. Now

$$AC = b - a, \qquad AM = a.$$

If angle CAM is α, then

$$cos\alpha = \frac{a}{b-a}.$$

The horizontal and vertical components of the forces give

$$R - S\ cos\alpha = 0, \qquad W - S\ sin\alpha = 0.$$

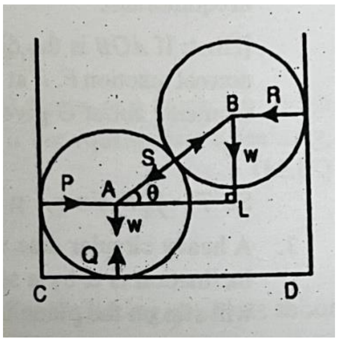
$$\therefore S = \frac{W}{sin\alpha}$$

$$\therefore R = W\ cot\alpha = W\frac{a}{\sqrt{(b-a)^2 - a^2}} = \frac{aW}{\sqrt{d^2 - 2ab}}.$$

From $R - S\ cos\alpha = 0, \frac{R}{S} = cos\alpha = \frac{a}{b-a} = \frac{a}{3a-a} = \frac{1}{2}.$

Example 15. Two equal smooth spheres, each of weight w and radius $r$are placed inside a smooth hollow cylinder, open at both ends, which rests on a smooth horizontal plane. Ifa $(< 2r)$ is the radius of the cylinder, find the reaction between the spheres and the cylinder. If W is the weight of the cylinder, show that the condition for the cylinder not to topple is

$$W > 2w\left(1 - \frac{r}{a}\right).$$



The figure is self-explanatory. The forces on the lower sphere arew, p, S, Qas in the figure. The horizontal components give $P = S\cos\theta$.

But the forces on the upper sphere are

$$\therefore \quad w, R, S.$$

$$S\cos\theta = R,$$

$$W = S\sin\theta.$$

$$\therefore \quad P = R = S\cos\theta = \frac{w}{\sin\theta}\cos\theta = w\cot\theta.$$

$$\therefore \quad S = \frac{w}{\sin\theta}, \qquad P = R = w\cot\theta.$$

But $AL = CD - 2r = 2a - 2r. AB = 2r$ and so

$$\cos\theta = \frac{AL}{AB} = \frac{a-r}{r}.$$

Thus S, P, R are known in terms of w, r, a.

If the cylinder topples, it will topple about D. Now the forces on the cylinder are

$$P, R, W, R_1.$$

Their distance from D are

$$r, r + AB\ sin\theta, a, 0.$$

Taking moments about D and using $P = R$,

$$r.P - (r + 2r\ sin\theta)\ R + aW\ >\ 0$$

$$i.e., \quad -2r\ sin\theta\ R + aW > 0 \quad or \quad aW > 2r\ sin\theta\ W\ cot\ \theta.$$

$$i.e., \quad aW > 2rWcos\theta \quad or \quad aW > 2rW\frac{a-r}{r}.$$

$$\therefore \quad W > 2w\left(1 - \frac{r}{a}\right).$$

Tilting of a body

When a gradually increasing force acts on a body which is in equilibrium, resting on a rough surface, the body either slips or tilts at some moment. This situation depends on the roughness of the surface, that is, the coefficient of friction (cof). If the cof is fairly small, slipping takes place and if the cof is fairly large, tilting takes place. For an intermediate particular value of cof, both slipping and tilting take place simultaneously. For any value of cof less than this particular value, the body slips and for any value of cof greater than this particular value the body tilts.

Now we shall consider physical situations in which the value of cof has the value such that the slipping and tilting take place simultaneously. In these cases the tilting is about a point of the rigid body and the forces at this point are

(i) The normal reaction R

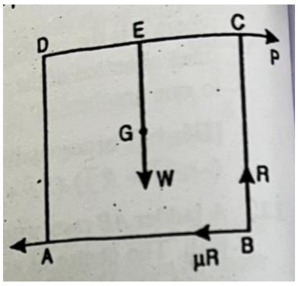
(ii) The frictional force μR in the direction opposite to that in which the body has a tendency to move.

The respective value of cof can be obtained by considering the components in two perpendicular directions and moments about a point in the preceeding equilibrium position.

EXAMPLES

Example 1. A rectangular thin plate $ABCD$ $(AB = a, BC = h)$ stands vertically with its side AB resting on a rough horizontal plane. A string attached to C is pulled horizontally with a gradually increasing force. Show that, if μ is the coefficient of friction, the plate will tilt if

$$\mu > \frac{a}{2h}.$$



Suppose that the plate will be in limiting equilibrium in a slightly tilted position with B alone being in contact with the horizontal plane whose coefficient of friction is μ. Then the forces acting on the plate will be

(i) Weight Wat G

(ii) Force Pat C

(iii)Normal reaction R and friction μR at μ.

Horizontal and vertical components and moments about B give

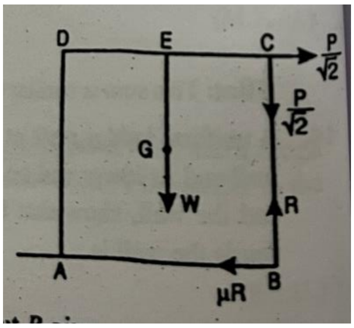
$$P = \mu R, R = W, \frac{a}{2}W = hP.$$

Eliminating Pand R, we get

$$\mu = \frac{a}{2h}.$$

So, if the cof is greater than $\frac{a}{2h}$, then the body tilts.

Example 2. A square lamina $ABCD$ of side $2a$ is standing upright on a horizontal plane with the side AB having contact with the plane. A string is attached to Cand pulled with aslowly increasing force P in a direction at right angles to the diagonal through the corner. Show that the lamina tilts if the coefficient of friction is greater than $\frac{1}{3}$.



Letμ be the cof. Now tilting takes place at B. The force Pat C has components

(i) $\frac{P}{\sqrt{2}}$horizontally and
(ii) $\frac{P}{\sqrt{2}}$vertically downward.

The other forces on the Lamina are

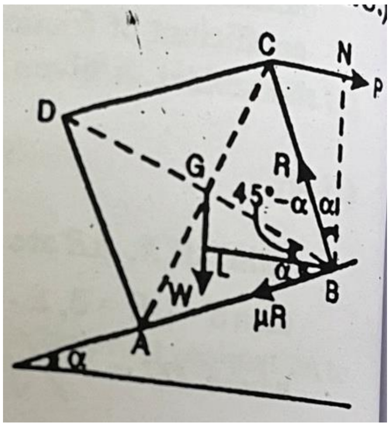
(i) Weight Wat G
(ii) Normal reaction R and friction μR at B

Thus the horizontal and the vertical components and moments about B give

$$\frac{P}{\sqrt{2}} = \mu R, R = W + \frac{P}{\sqrt{2}}, aW = \frac{2aP}{\sqrt{2}}.$$

Eliminating P and R, we get $\mu = \frac{1}{3}$. So, if the cof is greater than$\frac{1}{3}$ then the body tilts.

Example 3. ABCD is a uniform square plate of side $2a$ resting with its plane vertical with the side AB on a rough inclined plane of angle α. Cis the highest point of the plate. A gradually increasing force is applied at C horizontally. Prove that,λif is the angle of friction, then the plate tilts if $1 + \ tan\alpha < \ 2\ tan\ (\alpha + \lambda)$.



Suppose the plate will be in limiting equilibrium in a slightly tilted position standing on B on the inclined plane whose coefficient and angle of friction are μ and λ. Then the forces acting on the plate are

(i) μR along BA and R along BC at B
(ii) W vertically at G
(iii)P horizontally at C

Horizontal and vertical components give

$$P = R\ sin\alpha\ + \mu R\ cos\ \alpha,$$

$$W\ = R\ cos\alpha - \mu R\ sin\alpha.$$

$$\frac{P}{W} = \frac{sin\alpha + \mu cos\alpha}{cos\alpha - \mu sin\alpha} = \frac{sin\alpha + \tan\lambda\, cos\alpha}{cos\alpha - tan\lambda sin\alpha}.$$

$$= \frac{sin\alpha cos\lambda + cos\alpha sin\lambda}{cos\alpha cos\lambda - sin\alpha sin\lambda} = \frac{\sin(\alpha + \lambda)}{\cos(\alpha + \lambda)}$$

$$= \tan(\alpha + \lambda). \quad \ldots\ldots..(1)$$

If BL, BN are the perpendiculars drawn from B to the vertical through G and the horizontal through C, we get taking moments about B,

$$W.BL - P.BN = 0$$

$$or \quad \frac{P}{W} = \frac{BL}{BN} = \frac{\sqrt{2}a\cos(45° - \alpha)}{2a\ cos\alpha} = \frac{cos\alpha + sin\alpha}{2\ cos\alpha}$$

$$= \frac{1}{2}(1 + tan\alpha). \quad \ldots\ldots(2)$$

Equating (1) and (2), we get

$$\tan(\alpha + \lambda) = \frac{1}{2}(1 + tan\alpha)$$

This gives a value for λ. If the angle of friction is greater than this value, that is, if

$$\tan(\alpha + \lambda) > \frac{1}{2}(1 + tan\alpha)$$

then the body tilts.

Example 4: A solid cone of height h and base radius a is placed with its base on a rough inclined plane whose coefficient of friction is μ. The inclination of the plane is increased gradually. Show that the cone slips or topples according as

$$\mu < \frac{4a}{h} \ or \ \mu > \frac{4a}{h}$$

We shall assume that the cof μ is small so that, as α increases, the cone slips and does not topple. At the moment of slipping the forces on the cone are

$$R, \mu R, W$$

As in the figure. The corresponding along and perpendicular to the plane give

$$Wsin\alpha = \mu R, Wcos\alpha = R.$$

$$tan\alpha = \mu. \quad \dots \dots (1)$$

But as α increases from zero, $tan\alpha$ also increase from zero. So, when $tan\alpha$ attains the value μ, the cone slips.

Suppose the cofμ is large so that the cone tils about A and does not slips. At the moment of tilting the force acting on the cone are

$$R, F, W$$

As in the figure. So

$$R = Wcos\alpha.$$

Taking moments about O

$$OG \sin\alpha W - a\, R > 0$$

$$or \qquad \frac{h}{4} sin\alpha W - a\, W\, cos\alpha > 0$$

$$or \qquad tan\alpha > \frac{4a}{h}. \qquad \dots \dots . (2)$$

From (1) and (2) we get the critical values for $tan\alpha$ as

$$\mu, \frac{4a}{h}.$$

Thus, in conclusion, we have

(i) If $\mu < \frac{4a}{h}$, tan α attains the smaller value μ first and so the cone slips.

(ii) If $\mu > \frac{4a}{h}$, $tan\alpha$ attains the smaller value $\frac{4a}{h}$ first and so the cone slips.

Note:

If $\mu = \frac{4a}{h}$, slipping and tilting take place simultaneously when $tan\alpha$ attains the equal value.

EXERCISES

1. A sphere of weight W is held in equilibrium on a rough inclined plane of inclination α to the horizontal by a force P applied tangentially to its circumference and parallel to the plane. Show that $P = \frac{W}{2} sin\alpha$.
2. A heavy circular disc whose plane is vertical is kept at rest on a rough inclined plane whose inclination is α by a string parallel to the plane and touching the circle. Show that the disc will slip on the plane if the coefficient of friction is less than $\frac{1}{2} tan\alpha$.
3. A ladder is kept such that one end rests on a rough horizontal plane and the other end on a smooth vertical plane. Inclination of the ladder to the horizontal is θ. If a man of weight n times that of the ladder ascends it and if the ladder is just on the point of slipping when he reaches the top, calculate the coefficient of friction.

UNIT III

WORK, ENERGY AND POWER

3.1 WORK

A force is said to do work when its point of application moves. Let $\bar{F}$ be a force acting on a particle. When the particle moves, from a point whose position vector is $\bar{r}$, to a neighbouring point, whose position vector is $\bar{r} + \Delta\,\bar{r}$ the work done in this displacement of $\Delta\,\bar{r}$ is defined to be the scalar product

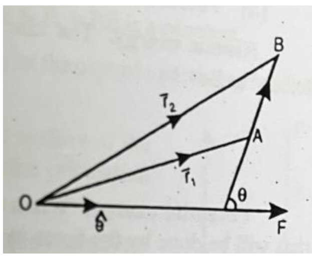
$$\bar{F}.\Delta\,\bar{r}$$

This scalar quantity is positive or negative according as the angle between $\bar{F}$ and $\Delta\,\bar{r}$ is acute or obtuse,the work done by a variable force $\bar{F}$, when the particle describes an arc C, is the line integral

$$\int_C \bar{F}.d\,\bar{r},$$

where $\bar{r}$ is the position vector of the particle.

Suppose that a constant force $F\hat{e}$ of magnitude F acts on a particle. When the particle moves from a point A whose position vector is $\bar{r_1}$ to a point B whose position vector is $\bar{r_2}$, the work done bythe force is



$$\int_{\bar{r_1}}^{\bar{r_2}} \bar{F}.d\,\bar{r} = \bar{F}.\int_{\bar{r_1}}^{\bar{r_2}} d\,\bar{r}$$

$$= \bar{F}.[\bar{r}]_{\bar{r_1}}^{\bar{r_2}}$$

$$= \bar{F}.(\bar{r_2} - \bar{r_1})$$

$$= \bar{F}.\overline{AB}$$

$$= F\ AB\ \cos\theta$$

$$= (\text{Mag. of const. force}) \times (\text{Dist. moved in the direction of the force}),$$

where θ is the angle between $e^{\wedge}$ and $\overline{AB}$.

3.1.1 Units of Work

The units of work in the different systems are given below:

System	Unit
M.K.S.	Joule
C.G.S.	Erg
F.P.S.	Foot-poundal

Joule: A joule is the work done by 1 newton in moving the particle through 1 metre.

Erg: An erg is the work done by 1 dyne in moving the particle through 1 cm. ($1\text{ joule} = 10^7\text{eeges}$).

Foot-Poundal: A foot-poundal is the work done by 1 poundal in moving the particle through 1 foot.

3.1.2 Work done in stretching an elastic string

Suppose OA is the natural length of a string. If the end O is fixed and the other end is pulled to B, we shall term

OB as the stretched length of the string and

AB as the extension due to pulling.

Bookwork 3.1. To show that the work done in stretching a string is

$$\text{Extension} \times \frac{(\text{Initial tension } + \text{ Final tension})}{2}$$

Consider an elastic string of natural length a whose coefficient of elasticity is λ. When the stretched length of the string is x, let the tension be T. Then, by Hooke's law,

$$T = \lambda \frac{\text{Extension}}{\text{Natural length}} = \lambda \frac{(x - a)}{a}$$

If the string is stretched further through an elementary distance dx, then the work done in this stretching is

$$T\,dx.$$

Therefore, the work W done in stretching the string from its stretched length b to the stretched length c is given by

$$W = \int_b^c T\,dx = \int_b^c \lambda \frac{(x - a)}{a}\,dx$$

$$= \frac{\lambda}{2a}[(x - a)^2]_b^c$$

$$= \frac{\lambda}{2a}[(c - a)^2 - (b - a)^2]$$

$$= \frac{\lambda}{2a}[(c - a) - (b - a)][(c - a) + (b - a)]$$

$$= \frac{\lambda}{2a}(c - b)[(c - a) + (b - a)]$$

$$= \frac{c - b}{2}\left[\frac{\lambda(b - a)}{a} + \frac{\lambda(c - a)}{a}\right]$$

$$= (c - b)\frac{T_1 + T_2}{2}$$

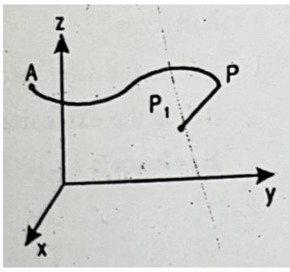
$$= \text{Extension} \times \frac{(\text{Initial tension} + \text{Final tension})}{2}.$$

3.2 CONSERVATIVE FIELD OF FORCE

Conservative forces: Certain forces have the property that the work done by them in adisplacement of the particle from one point to another depends only upon the initial and final positions of the particle and not on the nature of the paths pursued by them. Forces of this type are called conservative forces.

Conservative field of force: Let a force $\overline{F}$ be a vector point function of x, y, z.Then $\overline{F}(x, y, z)$ is said to define a force field. If $\overline{F}$ is conservative, then the force field is said to be conservative.

Bookwork 3.2. To show that in a conservative field of force, the force is the gradient of a scalar point function.



Suppose the symbol (AP) denotes an arc of a curve joining A and P. On it if E is a conservative force and $p(x, y, z)$ is a variable point and A, a fixed point (a, b, c), then the line integral

$$\int_{(AP)} \overline{F} . d\,\overline{r} \qquad \dots\dots\dots\dots\dots (1)$$

depends on $P(x, y, z)$ and not on the curve (AP). Hence the integral defines a scalar point function, say $\varphi(x, y, z)$ such that

$$\varphi(x, y, z) = \int_{(AP)} \overline{F} . d\,\overline{r} = \int_{(a,b,c)}^{(x,y,z)} \overline{F} . d\,\overline{r}$$

Further, let $P_1(x + \Delta x, y, z)$ be a point in the neighbourhood of P. Then

$$\varphi(x + \Delta x, y, z) = \int_{(AP_1)} \overline{F} . d\,\overline{r} = \int_{(AP)} \overline{F} . d\,\overline{r} + \int_{(PP_1)} \overline{F} . d\,\overline{r}$$

$$= \varphi(x, y, z) + \int_{(PP_1)} \overline{F} . d\,\overline{r}$$

$$\therefore \quad \varphi(x + \Delta x, y, z) - \varphi(x, y, z) = \int_{(PP_1)} \overline{F} . d\,\overline{r} \qquad \dots\dots\dots (2)$$

If (PP_1) is chosen as the straight-line joining P and P_1, then $(PP_1)$is parallel to the x axis and along it $dy = 0, dz = 0$, So, if $\bar{F} = F_1\bar{\imath} + F_2\bar{\jmath} + F_3\bar{k}$, then (2) becomes

$$\varphi(x + \Delta x, y, z) - \varphi(x, y, z) = \int_P^{P_1} (F_1\bar{\imath} + F_2\bar{\jmath} + F_3\bar{k}) . dx\,\bar{\imath}$$

$$= \int_P^{P_1} F_1\, dx = \int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1\, dx$$

$$\frac{\varphi(x + \Delta x, y, z) - \varphi(x, y, z)}{\Delta x} = \frac{1}{\Delta x}\int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1\, dx$$

Taking limit as $P_1 \to P$, that is, as $\Delta x \to 0$, we get

$$\frac{\partial\varphi}{\partial x} = F_1 \text{ and similarly } \frac{\partial\varphi}{\partial y} = F_2, \frac{\partial\varphi}{\partial z} = F_3$$

$$\bar{F} = \frac{\partial\varphi}{\partial x}\bar{\imath} + \frac{\partial\varphi}{\partial y}\bar{\jmath} + \frac{\partial\varphi}{\partial z}\bar{k} = \nabla\varphi$$

Remark 1: For a force field $\bar{F}$, if there exists a scalar function $\varphi(x, y, z)$ such that $\bar{F} = \nabla\varphi$, then $\bar{F}$ is conservative because

$$\int_C \bar{F} . d\,\bar{r} = \int_C \nabla\varphi . d\,\bar{r}$$

$$= \int_C \left(\frac{\partial\varphi}{\partial x}\bar{\imath} + \frac{\partial\varphi}{\partial y}\bar{\jmath} + \frac{\partial\varphi}{\partial z}\bar{k}\right) . (\bar{\imath}\, dx + \bar{\jmath}\, dy + \bar{k}\, dz)$$

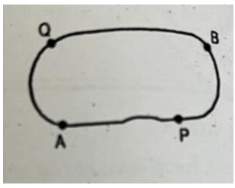
$$= \int_C \left(\frac{\partial\varphi}{\partial x}\, dx + \frac{\partial\varphi}{\partial y}\, dy + \frac{\partial\varphi}{\partial z}\, dz\right) = \int_C d\varphi$$

$$[\varphi]_{(a,b,c)}^{(x,y,z)} = \varphi(x, y, z) - \varphi(a, b, c)$$

which is independent of the configuration of C, where C is any curve joining A and P.

Remark 2. In a conservative field of force $\bar{F}$, if C is any simple closed curve, then

$$\int_C \bar{F} \cdot d\bar{r} = 0$$



For, if A, P, B, Q are points on C, then

$$\int_{C} = \int_{APB} + \int_{BQA} = \int_{APB} - \int_{AQB} = \int_{APB} - \int_{APB} = 0$$

Conversely, a force field $\overline{F}$ may be also defined to be conservative if the line integral of $\overline{F}$ along any simple closed curve vanishes.

In the cases of friction and resistance, the above line integral will always be negative and cannot vanish. So these forces are not conservative.

Remark 3. In a conservative field of force $\overline{F}, \nabla \times \overline{F} = \overline{0}$ because

$$\nabla \times \overline{F} = \nabla \times (\nabla\varphi) = 0.$$

3.2.1 Energy

Mechanical energy is of two kinds, namely,

1. Kinetic energy
2. Potential energy

Kinetic energy: The kinetic energy T of a particle of mass m and p. v. $\overline{r}$ at any moment is defined to be

$$T = \frac{1}{2} m \dot{\overline{r}} \cdot \dot{\overline{r}} = \frac{1}{2} m \overline{v} \cdot \overline{v} = \frac{1}{2} m v^2.$$

Potential energy: When a particle is subject to the action of a conservative force, the work that will be done by the force in the displacement of the particle from any point P to an arbitrarily chosen fixed point A is called the potential energy of the particle at P. Thus the potential energy V of the particle at P is

$$V = \int_{P}^{A} \overline{F} \cdot d\overline{r}.$$

Units of energy: Kinetic energy and potential energy have the same units as work.

3.2.2 Conservation of Energy

In the following bookwork, the first result is called principle of energy and the second result is called principle of conservation of energy.

Bookwork 3.3 : To show that, when a particle is subject to the action of conservative forces,

(i) the increase in K.E. in an interval is equal to the work done in that interval and

(ii) the sum of the K.E. and P.E. is a constant with respect to time.

(i) Let $\bar{r}_1, T_1$ and V_1 be respectively the position vector, K.E. and P.E. of a particle ofmass mat time t_1. Let $\bar{r}_2, T_2, V_2$ be the corresponding quantities at time t_2. Then

$$T_2 - T_1 = [T]_{T1}^{T2} = \int_{T1}^{T2} dT = \int_{t1}^{t2} \frac{dT}{dt} dt = \int_{t1}^{t2} \frac{d}{dt}\left(\frac{1}{2} m\dot{r}.\dot{r}\right) dt$$

$$= \frac{1}{2}\int_{t1}^{t2} m\,(\ddot{r}.\dot{r} + \dot{r}.\ddot{r})dt = \frac{1}{2}\int_{t1}^{t2} m\,(2\ddot{r}.\dot{r})dt = \int_{t1}^{t2} m\ddot{r}.\left(\frac{d\,\bar{r}}{dt}dt\right)$$

$$\int_{\overline{r1}}^{\overline{r2}} (m\ddot{r}).d\,\bar{r} = \int_{\overline{r1}}^{\overline{r2}} \bar{F}.\,d\,\bar{r}$$

$$= \text{workdone in the interval } t_2 - t_1$$

(ii) Choosing the position vector of the standard point for the calculation of the P.E. as $\overline{r_{0,}}$

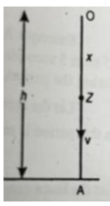
$$V_2 - V_1 = \int_{\overline{r2}}^{\overline{r0}} \bar{F}.d\,\bar{r} - \int_{\overline{r1}}^{\overline{r0}} \bar{F}.d\,\bar{r} = \int_{\overline{r2}}^{\overline{r0}} \bar{F}.d\,\bar{r} + \int_{\overline{r0}}^{\overline{r1}} \bar{F}.d\,\bar{r} = \int_{\overline{r2}}^{\overline{r1}} \bar{F}.d\,\bar{r}$$

$$= -\int_{\overline{r1}}^{\overline{r2}} \bar{F}.d\,\bar{r} = -(T_2 - T_1) = T_1 - T_2$$

$$T_1 + V_1 = T_2 + V_2$$

Thus $K.E. + P.E$ at $\bar{r_1}$ is equal to $K.E. + P.E$ at $\bar{r_2}$and consequently $K.E. + P.E$ is a constant.

Illustration 1: Verify the principle of conservation of energy in the case of a particle of mass m falling freely under gravity.



Consider a particle which starts from rest at O and hits the surface of the earth at A. Let $OA = h$. We shall choose A as the standard point for the calculation of the potential energy.

Let P be the position of the particle at time t and $OP = x$. Let the kinetic and potential energies of the particle at P be T, V. Then, by $v^2 = u^2 + 2as$,

$$T = ½\, mv^2 = ½\, m(2gx) = mgx$$

$$V = mg\,(AP) = mg\,(h - x)$$

$$T + V = mgx + mg(h - x) = mgh \quad \ldots..(2)$$

which shows that the mechanical energy of the particle is conserved.

Illustration 2: Verify the principle of conservation of energy sliding down on a smooth inclined plane freely under gravity.

Suppose a particle starts from rest at O on a smooth inclined plane OA. Let A be the standard position for the calculation of potential energy. If x is the distance travelled in time t and v, the velocity then,

$$T + V = ½\, mv^2 + (mg)(OA - x)\sin\alpha$$

$$= ½\, m(2g\sin\alpha\; x) + mg\sin\alpha\,(OA - x)$$

$$= mg\, OA\sin\alpha$$

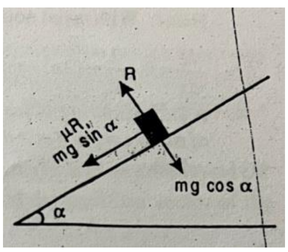
which is a constant. Therefore the mechanical energy of the particle is conserved.

EXAMPLES

1. A body is projected along a rough inclined plane straight up with kinetic energy E. Show that the work done against friction before the body comes to rest is

$$\frac{E\mu\cos\alpha}{\sin\alpha + \mu\cos\alpha,}$$

where μ is the coefficient of friction and α, the inclination of the plane to horizontal.



Let the velocity of projection be μ. Then

$$\frac{1}{2}\,mu^2 = E \quad or$$

$$u^2 = \frac{2E}{m}$$

Let the body move through a distance s before it comes to rest. Now the forces on the body up the plane are

(i) Component – mg sinα of the weight

(ii) Friction force -μR or $-\mu\, mg \cos\alpha$

So the acceleration of the body up the plane is

$$-(\sin\alpha + \mu\cos\alpha)g$$

Therefore, by $v^2 = u^2 + 2as$,

$$0 = \frac{2E}{m} - 2(\sin\alpha + \mu\cos\alpha)gs.$$

$$s = \frac{E}{m(\sin\alpha + \mu\cos\alpha)g}$$

Thus the work W done against the friction force is

$$W = (\text{Friction}) \times (\text{distance})$$

$$= \mu\, mg \cos\alpha \times \frac{E}{m(\sin\alpha + \mu\cos\alpha)g} = \frac{E\mu\cos\alpha}{\sin\alpha + \mu\cos\alpha}$$

3.3: POWER:

Power is the time rate of doing wok. The defining equation of power P is

$$P = \frac{dW}{dt},$$

Where W is the work done in time t. The units of power are watt and horse-power.

1 watt is 1 joule per second (1 kilowatt=100 watts)

1 horse power is 550 foot-pounds per second, that is, 550 g foot-poundals per second.

When the driving force F of an agent is a constant, the power developed then by the agent is obtained from "W = Fs" as

$$\frac{dW}{dt} = F\frac{ds}{dt} = Fv.$$

In other words, when the driving force F is a constant, the power developed at a moment is the force multiplied by the velocity at the moment.

EXAMPLES

Example 1 : Find the power of the pump which lifts 3000 litres of water per minute from a wall 10 meters deep and project it with a velocity of 16 m/sec. (1 litre water is of mass 1 kg.)

The pump work in two different ways, one in just carrying the water vertically upwards through 10m ant other in imparting to it a velocity of 16m/sec. when projected. Letthe work done in 1 sec .in these two ways be w_1 and w_2 joules. Now the mass of the water lifted in 1 sec. is

$$\frac{3000}{60}\text{kg. or } 50 \text{ kg}$$

Since the earth's attraction acting on this water is 50×9.8 newtons $=$ 490 netwons vertically downwards, the work done by gravitation in just lifting this water through 10 meters is

$$-490 \times 10\text{joules or} - 4900 \text{ joules}$$

But the work done by the pump is the negative of this. So

$$w_1 = 4900$$

The work done by the pump in 1 sec. in imparting a velocity to the water, is the increase in K.E. in 1 sec. which is

$$\frac{1}{2} \times 50 \times 16^2 - 0 \text{ joules or } w_2 = 6400 \text{ joules}$$

Thus the total work done by the pump in 1 sec. $w_1 + w_2$ joules $= 11300$ joules and hence the power of the pump is 11,300 watts or 11.3 kilowatts.

Example 2 : The total mass of a train is 500 tonnes. When it moves with a uniform speed of 72 km. p. h. on the level, the resistance due to friction is 16 kg-wt. per tonne. Find the power developed then.

The train moves with a constant velocity. So its acceleration is zero. So the drawing force ofthe engine just overcomes the resistance. Thus

$$\text{Driving force} = \text{Resistance} = 500 \times 16 \times 9.8\text{n}.$$

$$\text{const. velocity} = \frac{72 \times 1000}{60 \times 60} = 72 \times \frac{5}{18} = \frac{20\text{m}}{\text{sec}}.$$

$$\text{power} = (\text{Constant driving force}) \times (\text{velocity})$$

$$= (500 \times 16 \times 9.8) \times 20 = 15{,}68{,}000 \text{ watts}$$

$$= 1{,}568 \text{ kilowatts.}$$

Example 3 : A lift of mass 300kg rises from rest with uniform acceleration through a height of 15 m in 5 seconds. Find the power developed at the end of the fourth second and the average power during the period of the five seconds.

Let the acceleration of the lift be a m/sec^2. During the first five seconds. Since the initial velocity in this period is zero and the distance travelled is 15m., from $s = ut + \frac{1}{2}at^2$, we get

$$15 = \frac{1}{2}.\text{a}.5^2 \quad \text{or a} = 1.2.$$

Let the force exerted on the lift by the engine be F newtons. But the earth's attraction on the lift is 300×9.8 newtons. So the resultant force acting on it is

$$\text{F} - 300 \times 9.8 \text{ newtons}$$

So the equation of its motionma = F gives

$$300 \times 1.2 = \text{F} - 300 \times 9.8 \quad \text{or F} = 3{,}300$$

Since the velocity at the end of the fourth second is 4.8m/sec., from power = constant driving force × velocity, the rate of doing work at this moment is

$$3{,}300 \times 4.8 \text{joules /sec.} \quad \text{or} \quad 15{,}840 \text{ watts.}$$

Also the average power, namely, force × average velocity is

$$\text{F} \times \frac{\text{distance}}{\text{time}} = \frac{3{,}300}{5} \text{joules /sec.} = 9{,}900 \text{ watts.}$$

Example 4 : A car of mass 1 tonne attains a maximum speed 45km.p.h. when freely running down an incline of 1 in 10. What power must its engine develop to take it up an incline of 1 in 20 with the same speed if in both the cases the resistance is the same?

In the first case, the car moves freely with a constant velocity down the plane. So its acceleration is zero. So the gravitational force down the plane just overcomes the resistance. Thus

$$\text{Component of weight down the plane} = \text{Resistance R newtons}$$

$$1000\text{g} \times \frac{1}{10} = \text{R} \ \text{ or } \ \text{R} = 100\text{g} = 980.$$

In the second case, the car moves up the second plane with a constant velocity. So in acceleration is zero. So the driving force down the plane and

(i) The gravitational force down the plane and

(ii) The resistance R.

$$\text{F} = 100\text{g} \times \frac{1}{20} + \text{R} = 490 + 980 = 1470.$$

$$\text{power} = (\text{constant driving force}) \times (\text{velocity})$$

$$= 1470 \times \frac{45 \times 1000}{60 \times 60} = 18{,}375 \text{ watts.}$$

$$= 18.375 \text{ kilowatts.}$$

Example 5 : A train of mass 400 tonnes is ascending an incline of 1 in 200 with an acceleration of $\frac{1}{5}$ m/s^2. Find the frictional resistance of motion per tonne of the train when the speed is 20 km p.h.if the power developed then is 700 kilowatts.

The forces acting along the inclined plane are:

(i) The driving force F newtons up the plane

(ii) The resistance R newtons down the plane

(iii) The component along the plane in the downward direction of the weight of the train, namely,

$$400 \times 1000 \times 9.8 \times (1/200) = 19600 \text{ N}$$

Hence the resultant force up the plane is: $\text{F} - \text{R} - 19600$

Thus, from $\text{ma} = \text{F}$: $400 \times 1000 \times \left(\frac{1}{5}\right) = \text{F} - \text{R} - 19600$ or

$$\text{F} = \text{R} + 80000 + 19600 = \text{R} + 99600$$

But we know that: $\text{Power} = \text{driving force} \times \text{velocity}$. Hence the power developed is 700000 watts.

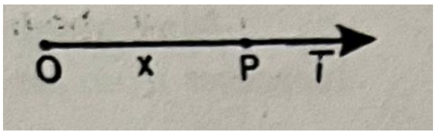
$$700000 = (R + 99600) \times \frac{20 \times 1000}{60 \times 60} \quad \text{or } R = 26400 \text{ N.}$$

The resistance per tonne is: $26400 / 400 = 66$ N

3.4 SIMPLE HARMONIC MOTION

The motion of a particle moving along a straight line with an acceleration which is always towards a fixed point on the straight line and whose magnitude is proportional to the distance of the particle from the fixed point is called a *simple harmonic motion*. First we shall find the equation of a simple harmonic motion.

Equation of motion:



Let O be the fixed point, P the position of the particle at any time t and OP$= x$.Let $\bar{\imath}$ be the unit vector in the direction as shown in the figure and the position vector of the particle with reference to O at time t be $\bar{r}$. Then

$$\bar{r} = x\bar{\imath}. \qquad \therefore \ \ddot{\bar{r}} = \ddot{x}\,\bar{\imath}.$$

The acceleration is proportional to the distance from O. So its magnitude can be taken as n^2x, where n^2 is a positive constant. Further, the acceleration is towards O. Hence it is

$$n^2x(-\bar{\imath}).$$

Equating the above two quantities, we get

$$\ddot{x}\bar{\imath} = n^2x(-\bar{\imath}).$$

This is the equation of motion of the particle. The scalar from of this equation is

$$\ddot{x} = -n^2x.$$

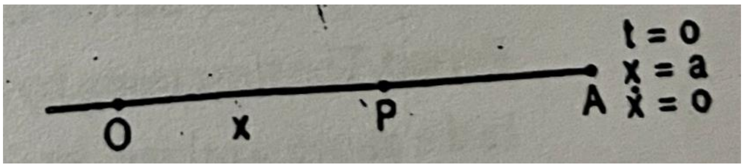
Harmonic motion: The term *harmonic motion* without the qualifier "simple" denotes a general motion in which the variable x is not a displacement along a straight line but an angle θ or an arcual distance, etc., such that

$$\ddot{\theta} = -n^2\theta, \ \ddot{s} = -n^2s.$$

Bookwork 3.4 : In the S.H.M. whose equation is

$$\ddot{x} = -n^2 x$$

to express (i) x in t (ii) $\dot{x}$ in t (iii) $\dot{x}$ in x.



Let O be the fixed point towards which the acceleration is and let the particle be at rest initially at A, where $OA = a$. The equation of motion may be rewritten as

$$(D^2 + n^2)x = 0.$$

Its auxiliary equation is $m^2 + n^2 = 0$ and its roots are in and $-in$. So the general solution of the equation of motion is

$$x = A\cos nt + B\sin nt. \qquad \text{..........} (1)$$

From the initial conditions, when $t = 0$, x = a. So

$$a = A\cos n(0) + B\sin n(0) \quad \text{or} \quad A = a.$$

Thus (1) becomes

$$x = a\cos nt + B\sin nt. \qquad \text{...........} (2)$$

$$\therefore \qquad \dot{x} = -\text{an}\sin \text{nt} + \text{Bn}\cos \text{nt} \qquad \text{...........} (3)$$

But, when $t = 0$, $v = \dot{x} = 0$. Therefore

$$0 = -\text{an}\sin \text{n}(0) + \text{Bn}\cos \text{n}(0) \quad \text{or} \quad B = 0.$$

Thus (2) and (3) become

$$x = a\cos nt, \qquad \text{..........} (4)$$

$$\dot{x} = -\text{an}\sin \text{nt} \quad \text{or} \quad v = -\text{an}\sin \text{nt} \qquad \text{...........} (5)$$

Eliminating t from (4), (5), we get

$$\frac{x^2}{a^2} + \frac{v^2}{a^2 n^2} = 1 \ \ or \ \ v^2 = n^2(a^2 - x^2) \qquad \text{...........} (6)$$

Now (4), (5), (6) are the required results.

Maximum speed: It is clear that, as t increases from zero, x decreases from a and the speed increases from zero. The speed is maximum when $x = 0$. This maximum speed is na.

Nature of motion: The particle has same speed at points equidistant from O. So the motion is an oscillatory motion between A and A′, with O as the mean position, where $O\grave{A} = OA$.

Other defining equations of S.H.M: If the velocity $\dot{x}$ of a particle moving in a straight line is given by $\dot{x}^2 = n^2(a^2 - x^2)$, then its motion is simple harmonic because differentiation of this with respect to t gives

$$2\dot{x}\ddot{x} = -2n^2 x\dot{x} \quad or \quad \ddot{x} = -n^2 x.$$

Also, if the displacement of the particle is given by $x = a\cos(nt + \varepsilon)$, then also the motion is simple harmonic. In this case

$$x = -n\,a\cos(nt + \varepsilon), \; or \quad \ddot{x} = -n^2 a\cos(nt + \varepsilon) = -n^2 x.$$

Also the equations

$$x = a\cos(nt + \varepsilon), \; x = A\cos nt + B\sin nt$$

define S.H.M.'s.

Now we furnish a set of definitions pertaining to a simple harmonic motion.

Oscillation: One complete motion of the particle from a point on its path to one extremity of its path, then to the other extremity and back to the point, is called an oscillation.

Vibration: The motion of the particle from one extremity to the other extremity of its path, is called a vibration.

Amplitude: The maximum distance through which the particle moves on either side of the mean position of the motion, is called the amplitude of the motion ($OA = a$ is the amplitude). In an oscillation the particle travels along a distance equal to 4 times amplitude.

Period: The time taken by the particle to make one oscillation is called the period of the motion.

In the above working, let the time taken by the particle to move from A to O be t_0. Then, from (4),

$$0 = a\cos nt_0 \quad or \;\; nt_0 = \frac{\pi}{2} \quad or \quad t_0 = \frac{\pi}{2n}.$$

But the period T is four times t_0. Therefore

$$T = 4.\frac{\pi}{2n} = \frac{2\pi}{n}.$$

It is important to note that the period of a simple harmonic motion is independent of its amplitude.

Frequency: The number of oscillations per second is called the frequency of the motion; that is, the frequency is the reciprocal of the period. So it is

$$\frac{1}{T} \ or \ \frac{n}{2\pi}.$$

Phase and epoch: The general form of the displacement x of the particle is

$$x = a \cos{(nt + \varepsilon)}.$$

Here $nt + \varepsilon$ is called the *phase* at time t. The initial phase, that is, the phase when $t = 0$, is called *epoch*. So ε is the epoch.

Bookwork 3.5 : To show that, in a simple harmonic motion, the sum of the K.E. and P.E. is constant.

$$K.E \ at \ P = \frac{1}{2}m(vel.)^2 = \frac{1}{2}m[n^2(a^2 - x^2)]$$

Taking O as the standard point for the calculation of P.E.,

$$P.E \ at \ P = \int_p^0 (force)dx = \int_p^0 (m\ddot{x})dx$$

$$= \int_p^0 m(-n^2x)dx = \frac{1}{2}mn^2x^2.$$

$\therefore$ $$K.E + P.E \ = \frac{1}{2}mn^2(a^2 - x^2) + \frac{1}{2}mn^2x^2 = \frac{1}{2}mn^2a^2$$

which is a constant.

Bookwork 3.6 : If initially the particle is projected from A with a velocity V away from O (OA= a) then to find the S.H.M.

Now the initial conditions are when

$$t = 0, x = a, v = \dot{x} = V.$$

So in $x \ = \ C \cos nt \ + \ D \sin nt, \ C = a, D = \frac{V}{n}.$

$\therefore \qquad x = a\cos nt + \frac{V}{n}\sin nt, \frac{\dot{x}}{n} = -a\sin nt + \frac{V}{n}\cos nt.$

Squaring and adding these two, we get

$$x^2 + \frac{v^2}{n^2} = a^2 + \frac{v^2}{n^2}.$$

or $$v^2 = n^2(a^2 - x^2) + V^2.$$

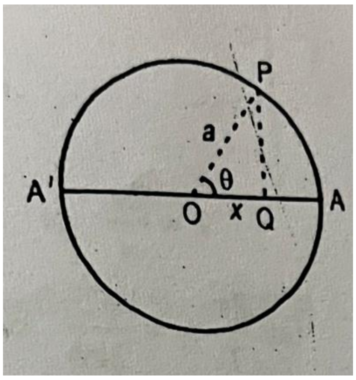
Amplitude: The amplitude of this motion is the value of x when $\dot{x} = 0$. So it is

$$\sqrt{a^2 + \frac{v^2}{n^2}}.$$

3.4.1 Projection of a particle having uniform circular motion.

In this section we consider a circular motion whose projection on a diameter illustrates a simple harmonic motion.

Bookwork 3.7 : A particle moves along a circle with a uniform speed. To show that the motion of its projection on fixed diameter is simple harmonic.



Let us have the following assumptions:

O : Centre of the circle

a : Radius of the circle

AÁ : A diameter of the circle

P : Position of the particle at time t

Q : Projection of P on AÁ

θ : Angle AOP

ω : Angular velocity $\dot{\theta}$ of P about O which is a constant

Let the distance of Q, the projection of P, from O be x. Then

$$x = OQ = acos\theta.$$

$$\dot{x} = -\, a\, sin\theta\, \dot{\theta} \;= -a\omega sin\theta,$$

$$\ddot{x} = -a\omega\, cos\theta\, \dot{\theta} \;= -a\omega^2\, cos\theta$$

$$= -\omega^2 x.$$

So the motion of Q along the diameter is simple harmonic whose amplitude and period of oscillation are

$$a, \frac{2\pi}{\omega}.$$

Remark: Suppose initially P commences its uniform motion when $\theta = \varepsilon$. Then, at time t,

$$\theta = \omega t + \varepsilon.$$

Therefore, at time t,

$$x \;=\; a\, cos(\omega t + \varepsilon).$$

Thus, in this case, at time t, the phase is the angle $\omega t + \varepsilon$ and the epoch is ε.

3.4.2 Composition of two simple harmonic motions of same period.

In this section we study the superposition of two simple harmonic motions.

Bookwork 3.8 : To show that the resultant of two simple harmonic motions of *same period* along the same straight line is also simple harmonic with the same period.

Let the displacements in the two given motions be

$$x_1 \;=\; a_1\, cos(nt + \varepsilon_1),\; x_2 \;=\; a_2\, cos(nt + \varepsilon_2)$$

Then the resultant displacement x is given by

$$x = x_1 + x_2 = a_1 cos(nt + \varepsilon_1) + a_2 cos(nt + \varepsilon_2)$$

$$= a_1(cos\, nt\, cos\, \varepsilon_1 - sin\, nt\, sin\, \varepsilon_1) + a_2(cos\, nt\, cos\, \varepsilon_2 - sin\, nt\, sin\, \varepsilon_2)$$

$$= (a_1 cos\, \varepsilon_1 + a_2 cos\, \varepsilon_2)\, cos\, nt - (a_1 sin\, \varepsilon_1 + a_2 sin\, \varepsilon_2)\, sin\, nt$$

$$= (a \cos \varepsilon) \cos nt - (a \sin \varepsilon) \sin nt$$

$$= a \cos(nt + \varepsilon),$$

where

$$a \cos \varepsilon = a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2, a \sin \varepsilon = a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2$$

$$a = \sqrt{(a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2)^2 + (a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2)^2}$$

$$= \sqrt{a_1^2 + a_1^2 + 2a_1 a_2 \cos(\varepsilon_1 - \varepsilon_2)}$$

and $$\varepsilon = \tan^{-1} \frac{a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2}{a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2}.$$

So the resultant motion is also simple harmonic with the same period as the component motions. Its amplitude is a and epoch is ε.

Bookmark 3.9 : To show that the resultant motion of two simple harmonic motions of same period along two perpendicular lines, is along an ellipse.

Choose the lines of motion as the x, y axes. Let us count the time from the moment when the first particle is at one extreme of its path, so that at time t, its displacement is

$$x = a \cos nt.$$

Let the displacement of the second particle, at that time t, be

$$y = b\cos(nt + \varepsilon)$$

$$= b(\cos nt \cos \varepsilon - \sin nt \sin \varepsilon).$$

Elimination of t from these two equations gives the equation of the path, corresponding to the resultant motion, as

$$y = b\left(\frac{x}{a}\cos\varepsilon \pm \sqrt{1 - \frac{x^2}{a^2}}\,\sin\varepsilon\right)$$

or $$\left(\frac{y}{b} - \frac{x}{a}\cos \varepsilon\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\sin^2\varepsilon$$

or $$\frac{x^2}{a^2} - 2\cos \varepsilon \frac{xy}{ab} + \frac{y^2}{a^2} = \sin^2\varepsilon$$

which is an ellipse since the equation satisfies "$h^2 - ab < 0$".

EXAMPLES

Example 1 : The displacement x of a particle moving along a straight line is given by $x= A\,cos\,nt\; +\; B\,sin\,nt, where\; A, B, n\; are\; constants$ Show that its motion is S.H. If A= $3, B = 4, n = 2$, find its period, amplitude, maximum velocity and maximum acceleration.

Differentiating $x= A\,cos\,nt\; +\; B\,sin\,nt,$ twice w. r. t. t, we get

$$\ddot{x} = -n^2 x.$$

so the motion is simple harmonic.

When A= $3, B = 4, n = 2,$

$$x = 3\,cos\,2t + 4\,sin\,2t$$

$$v = \dot{x} = 2(-3\,sin\,2t + 4\,cos\,2t).$$

$$\therefore \qquad x^2 + \left(\frac{v}{2}\right)^2 = 3^2 + 4^2$$

When $v = 0, x =$ amplitude $a = \sqrt{3^2 + 4^2} = 5.$

Max. velocity $= na = 2 \times 5 = 10,$

Max. acceleration $n^2 a = 4 \times 5 = 20.$

Example 2 : A particle executes S.H.M. and while moving from the mean position to one extreme position its distances at three consecutive seconds are x_1, x_2, x_3. Show that its period is

$$\frac{2\pi}{cos^{-1}\{(x_1+x_2)/2x_2\}}.$$

Let the three consecutive seconds be $t-1, t, t+1$. Then, from "$x = a\,cos\,nt$",

$$x_1 = a_1\,cos\,n(t-1),\; x_2 = a\,cos\,nt,\; x_3 = a\,cos n(t+1).$$

$$\therefore \qquad x_1 + x_3 = a\{cos\,(nt-n) + cos\,(nt+n)\}$$

$$= a\{2\,cos\,nt\,cos\,n\} = 2x_2 cos\,n$$

$\therefore \qquad \dfrac{x_1 + x_3}{2x_2} = cos n \quad or \quad n = \quad cos^{-1}\dfrac{x_1 + x_3}{2x_2}.$

But the period is $\frac{2\pi}{n}$. So the result, as stated.

Example 3 : A particle executing a S.H.M. with O as the mean position and a as the amplitude. When it is at a distance $\frac{a}{2}$ from O, its velocity is quadrupled by a blow. Show that its new amplitude is $\frac{7a}{2}$.

Let v and 4v be the velocities immediately before and after the blow and a_1, the new amplitude.Then from "$x^2 = n^2(a^2 - x^2)$"

$$v^2 = n^2\left[a^2 - \frac{a^2}{4}\right], \qquad \text{………..} (1)$$

$$16v^2 = n^2\left[a_1^2 - \frac{a^2}{4}\right], \qquad \text{……….} (2)$$

Eliminating v^2 from (1) and (2),

$$16\left[\frac{3a^2}{4}\right] = n^2\left[a_1^2 - \frac{a^2}{4}\right] \quad \text{or} \quad 12a^2 = a_1^2 - \frac{a^2}{4}$$

$\therefore \qquad \dfrac{49a^2}{4} = a_1^2 \quad or \quad a_1 = \dfrac{7a}{2}.$

Example 4 : A particle is executing a S.H.M. of period T with O as the mean position. The particle passes through a point P with velocity V in the direction of OP. Show that the time which lapses before its return to P is

$$\frac{T}{\pi} tan^{-1}\frac{VT}{2\pi\, OP}.$$

Let the particle take a time t_1 to reach the end A from P. Then the time it takes to reach P from A is also t_1. Now the required time is $2t_1$. Let OP = b and OA = a. Then, considering the motion from A to P, from $x = a\cos nt$ and $v^2 = n^2(a^2 - x^2)$,

b = a cos nt_1 , $V^2 = n^2(a^2 - x^2)$.

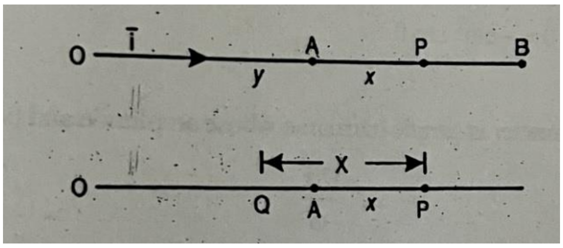
$\therefore \qquad \cos nt_1 = \dfrac{b}{a} \quad or \quad \tan nt_1 = \dfrac{\sqrt{a^2 - b^2}}{nb} = \dfrac{V}{nb}.$

But $T = \frac{2\pi}{n}$ or $n = \frac{2\pi}{T}$. Therefore,

$$tan\frac{2\pi}{n}t_1 = \frac{VT}{2\pi b} \quad or \quad \frac{2\pi t_1}{T} = tan^{-1}\frac{VT}{2\pi b}.$$

$$\therefore \qquad 2t_1 = \frac{T}{\pi}tan^{-1}\frac{VT}{2\pi\, OP}.$$

Example 5 : A particle P of mass m moves in a straight line OB under the force $mn^2 \times$ (distance from A) directed towards A, where A moves along OB with constant acceleration α. Show that the motion of P is simple harmonic, of period $\frac{2\pi}{n}$, about a moving centre which is always at a distance $\frac{\alpha}{n^2}$ behind A.



Let $\bar{\imath}$ be the unit vector along the line, OA = y and AP = x. Then the position vector of P with respect to the fixed point O, is

$$\bar{r} = (x + y)\bar{\imath}.$$

The force acting on the particle is

$$\bar{F} = mn^2\, AP(-\bar{\imath}) = -mn^2 x\bar{\imath}\,.$$

So, the equation of motion, $m\,\ddot{\bar{r}} = \bar{F}$, is

$$m\,(\ddot{x} + \ddot{y})\,\bar{\imath} = -mn^2 x\bar{\imath} \quad or \quad \ddot{x} + \ddot{y} = -n^2 x.$$

But $\ddot{y} = \alpha$. Hence

$$\ddot{x} + \alpha = -n^2 x \;\; or \;\; \ddot{x} = -n^2\left(x + \frac{\alpha}{n^2}\right).$$

Set $x + \frac{\alpha}{n^2} = X$. Then X is the distance of P from a point Q behind A at a distance $\frac{\alpha}{n^2}$ and

$$\ddot{X} = -n^2 X$$

which shows that the motion of P is S.H. with period $\frac{2\pi}{n}$ and with mean position Q.

Example 6 : A horizontal shelf oscillates vertically with a S.H.M. of period $\frac{2\pi}{n}$ and amplitude a. Show that a book of mass m resting on the shelf will not leave it provided

$n^2 < g/a.$ In the case where it leaves, obtain its velocity then.

Let us make the following assumptions:

O	:	Mean position of S.H.M.
A, A′	:	Highest and lowest points on the path
$\bar{j}$	:	Unit vector in the vertically upward direction
P	:	Position of the book at time t
x	:	OP
$\bar{r}$	:	Position vector x$\bar{d}$ of P with reference to O

The forces acting on the book are earth's gravitation $-$mg $\bar{j}$ and reaction of the shelf R $\bar{j}$. So the equation $m\ddot{\bar{r}} = \bar{F}$ of motion of the book is

$$m\,\ddot{x}\bar{j} = -mg\,\bar{j} + \mathrm{R}\,\bar{j}$$

or
$$m\,\ddot{x} = R - mg \qquad \text{……... (1)}$$

The period of the S.H.M. of the shelf is $\frac{2\pi}{n}$. So its equation is

$$\ddot{x} = -n^2x. \qquad \text{……… (2)}$$

Thus, eliminating $\ddot{x}$ from (1) and (2),

$$R = m(g - n^2x).$$

This shows that as the shelf moves from O to A, that is, as x increases from zero, R decreases gradually. But, at A, $x = a$. So, if $n^2a = g$, that is, if $n = \sqrt{g/a}$, then R vanishes at A but immediately becomes positive thereafter. So the book does not lose its contact with the shelf. If $n > \sqrt{g/a}$, then R vanishes at a point between O and A and there the book loses its contact with the shelf. At this moment the value of x is given by

$$g - n^2x = 0 \quad as \quad x = g/n^2.$$

The velocity v of the book then is obtained from $\dot{x}^2 = n^2(a^2 - x^2)$ as follows:

$$v^2 = n^2\left(a^2 - \frac{g^2}{n^4}\right) = \frac{n^4a^2 - g^2}{n^2} \quad \text{or} \quad v = \frac{1}{n}\sqrt{n^4a^2 - g^2}.$$

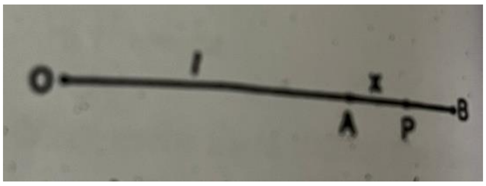
R does not vanish at all when the shelf moves from A′ to O because, in this motion, x is negative and hence R is positive.

3.5 S.H.M. ALONG A HORIZONTAL LINE

In this section we consider simple harmonic motions along a horizontal line, the forces being tensions of light elastic strings or light spiral springs. Here, by Hooke's law,

$$Tension = \lambda \frac{Extension\ or\ compression}{Natural\ length}.$$

Bookwork 3.10 : One end of a light spiral spring of length l is fixed to a fixed point$O$on a smooth horizontal table and a heavy particle of mass m is attached to the other end. If the particle is pulled through a distance $a,$ $(a < l),$ and then let go, to find its motion.



Let $OA = l$ and $AB = a$. Let P be the position of the particle at time t after the particle is let go. Let the direction OA be the positive direction to measure the distance and $AP = x$. Then the acceleration of the particle in this horizontal direction is $\ddot{x}$ and the tension on it is in the opposite direction with a magnitude

$$\lambda \frac{x}{l}.$$

So the corresponding equation of motion is

$$m\,\ddot{x} = -\lambda \frac{x}{l}$$

If we denote the positive constant $\frac{\lambda}{lm}$ by n^2, then

$$\ddot{x} = -n^2 x.$$

This shows that the particle executes a S.H.M. such that

i) $x = a\cos nt = a\cos\sqrt{\frac{\lambda}{lm}}t$

ii) $v^2 = n^2(a^2 - x^2) = \frac{\lambda}{lm}(a^2 - x^2)$

iii) $T = \frac{2\pi}{n} = 2\pi\sqrt{\frac{\lambda}{lm}}.$

Remark: If the spring is replaced with an elastic string and $\acute{A}, \acute{B}$, are points such that $O\acute{A} = OA$, $O\acute{B} = OB$, then the motion of the particle will be S.H. so long as the tension acts; otherwise, it will be a free motion under no force. Thus the motions from

B to A, $\acute{A}$ to $\acute{B}$, $\acute{B}$ to $\acute{A}$, A to B

Form a complete oscillation of the S.H.M. and the motions from A to $\acute{A}$ and $\acute{A}$ to A are free motions with a constant speed. As such, the time taken for one full to-and-fro motion, from B to $\acute{B}$ and back to B, is the period of the S.H.M. plus twice the time for the motion from A to $\acute{A}$. So it is

$$\frac{2\pi}{n} + 2\frac{A\acute{A}}{maximum\ velocity} = \frac{2\pi}{n} + 2.\frac{2l}{na} = \frac{l}{n}\left(2\pi + \frac{4l}{a}\right)$$

$$= 2\sqrt{\frac{lm}{\lambda}}\left(\pi + \frac{2l}{a}\right)$$

EXAMPLES

Example 1 : The ends of an elastic string of natural length *a* are fixed at points A and B, distance 2*a* apart, on a smooth horizontal table. A particle of mass *m* is attached to the middle point of the string and slightly displaced along the direction perpendicular to AB. Show that the period of small oscillation is

$$\pi\sqrt{\frac{2am}{\lambda}}.$$

Let O be the midpoint of AB, P the position of the particle at time t, and $OP = x$. The force acting on m along PO and towards O is

$$2T\cos APO.$$

$$m\ddot{x} = -2T\cos APO = -2\lambda \frac{Extension}{nat.\ length}\frac{x}{AP}$$

$$= -2\left(\lambda \frac{2AP-a}{a}\right)\frac{x}{AP} = -\frac{2\lambda x}{a}\left(2-\frac{a}{AP}\right)$$

$$= -\frac{2\lambda x}{a}\left(2-\frac{a}{\sqrt{a^2+x^2}}\right).$$

For a small oscillation, x is small. So

$$2-\frac{a}{\sqrt{a^2+x^2}} = 2-1 = 1.$$

$\therefore \qquad m\ddot{x} = -\frac{2\lambda x}{a} \quad or \quad \ddot{x} = -\frac{2\lambda}{a}x.$

$\therefore \qquad$ Period$= 2\pi\sqrt{\frac{am}{2\lambda}} = \pi\sqrt{\frac{2am}{\lambda}}.$

3.6 S.H.M. ALONG A VERTICAL LINE

In this section we consider simple harmonic motions along a vertical line, the forces being tensions of light elastic strings or light spiral springs and earth's gravitation.

Bookwork 3.11 : A light spiral spring of length l hangs vertically in the position OA where O is the point of suspension. Let OB be its equilibrium position when a mass m is attached to its lower end and AB$= a$. If m is pulled vertically downward from B to C, through a distance b, and let go, to find its motion.

The forces on the particle in the equilibrium position are

(i) Weight mg vertically downward

(ii) Tension T of the string upwards

with their magnitudes equal. Therefore,

$mg = \lambda\frac{a}{l}.$ .... (1)

Let us choose the downward vertical direction as the positive direction to measure the distance. Let P be the position of the particle at time t. Let its distance from the equilibrium position B be x. Then the acceleration in the positive direction is $\ddot{x}$ and the forces in this direction are

(i) The weight mg

(ii) The tension $-\lambda\frac{a+x}{l}$.

$$\therefore \qquad m\ddot{x} = mg - \lambda\frac{a+x}{l} = mg - \lambda\frac{a}{l} - \frac{\lambda}{l}x$$

$$= -\frac{\lambda}{l}x by (1)$$

$$\therefore \qquad \ddot{x} = -\frac{\lambda}{lm}x = -n^2x, \ say.$$

So the motion is simple harmonic with B as its mean position. Its amplitude is b because at C (BC$= b$) its velocity is zero. Its period and maximum speed are

$\frac{2\pi}{n}, nb.$

Remark: Suppose that the spring is replaced with an elastic string. Then the motion of the string from C to A will be S.H. and the motion above A will be a free motion under earth's gravity till the string becomes slack.

EXAMPLE

Two bodies of masses m and $\acute{m}$ are attached to the lower end of an elastic string whose upper end is fixed and hang at rest. $\acute{m}$, falls off. Show that the distance of m from the upper end of the string at time t is

$$a + b + c \cos\sqrt{\frac{g}{b}}\ t.$$

where a is the unstretched length of the string and b and c are the distances by which it would be stretched when supporting m and $\acute{m}$.

Let OA denote the natural length a of the string and B the equilibrium position of m so that $AB = b$. When m is in equilibrium the forces on it in the downward direction are

$mg, -\lambda\frac{b}{a}.$

Their magnitudes are equal.

$$\therefore \qquad mg = \lambda\frac{b}{a} \qquad \text{……….. (1)}$$

Similarly, for $\acute{m}$,

$$\acute{m}g = \lambda\frac{c}{a}. \qquad \ldots\ldots\ldots\ldots (2)$$

Therefore, from (1) and (2),

$$(m+\acute{m})g = \lambda\frac{b+c}{a}.$$

That is, if C is the equilibrium position of $m+\acute{m}$, then

$$AC = b+c,\ BC = c.$$

Let P be the position of *m* at time *t* after $\acute{m}$ falls off from C and BP= x. . Then

$$m\ddot{x} = mg - \lambda\frac{b+c}{a} = mg - \frac{\lambda b}{a} - \frac{\lambda x}{a}.$$

$$\therefore \qquad m\ddot{x} = -\frac{\lambda x}{a} \quad by\ (1)$$

$$\therefore \qquad \ddot{x} = \frac{-\lambda}{am}x.$$

Thus the motion of *m* is S.H. with B as the mean position. The amplitude is

BC= c because the velocity at C is zero. Therefore

$$x = c\cos\sqrt{\frac{\lambda}{am}}t = c\cos\sqrt{\frac{g}{b}}t\ \ by\ (1).$$

$$\therefore \qquad Op = OA + AB + x = a + b + c\cos\sqrt{\frac{g}{b}}t.$$

UNIT 4

PROJECTILE

4.1 FORCES ON A PROJECTILES

In this chapter we study coplanar motions of particles under gravity only, neglecting the air resistance and assuming that the motion is closed the surfaces to the earth so that g is a constant.

Projectile. A particle or body projected is called a projectile.

Trajectory. The path pursed by a projectile is called the trajectory of the projectile.

Horizontal range. If O is the point of projection and if A is the point at which the projectile hits the horizontal plane through O, then OA is called the horizontal range.

Range on an inclined plane. Suppose OA is a line of greatest slope on an inclined plane. If a particle projected from O hits on OA at B, then OB is the range on the inclined plane.

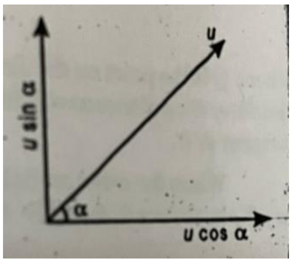
4.1.1 Displacement as a combination of vertical and horizontal displacements

The only force acting on a projectile is the earth's gravity which is always vertically downward. So the projectile has an acceleration g always in the vertically downward direction. Now we shall consider the displacement of the projectile as a resultant of the displacements in two perpendicular directions, namely,

(i) Horizontal direction

(ii) Vertical direction

(i) **Horizontal displacement.** In the horizontal direction there is no force on the projectile. So the acceleration of the projectile in this direction is zero and hence the projectile moves in the horizontal direction with a constant velocity which is the component of the velocity of projection in this direction.



If the projectile is projected with a velocity u in a direction which is inclined to the horizontal at an angle α, then its horizontal component of velocity is

$$u\ cos\alpha$$

So the horizontal displacement of the projectile in time t is

$$(velocity) \times (time)$$

or$(u\ cos\alpha)$t.

(ii) Vertical displacement. To study the vertical displacement we shall choose the upward direction as the positive direction for measuring the distance of the particle. In this direction the acceleration of the particle is –g. So the vertical displacement can be studied with the equations meant for constant accelerations a, namely,

$$v = u + at,$$

$$s = ut + \frac{1}{2}at^2,$$

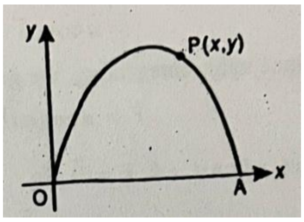
$$v^2 = u^2 + 2as.$$

Now $a = -g$ and the initial velocity in the positive direction is

$$u\ sin\ \alpha.$$

4.1.2 Nature of trajectory

Bookmark 4.1 To show that the path of a projectile is a parabola.



Suppose a particle of mass m is projected from a point O with a speed u in a direction which makes an angle α with the horizontal. Let the particle hit the horizontal plane through O, at A. Choose OA as the axis and the upward vertical line through O as the y axis. Let

$$p = (x, y)$$

be the position of the particle at time t. Then the horizontal and vertical displacements of the particle at time t are

$$(x, y)$$

Since earth's gravity is the only force on the projectile the horizontal and upward vertical components of the acceleration are

$$0, -g$$

Therefore the horizontal component of the velocity is always a constant which is the initial component$u\ cos\ \alpha$. Therefore the horizontal displacement in time t is

$$x = (\text{velocity}) \times (\text{time}) = (u\ cos\ \alpha)t. \qquad ......(1)$$

For the upward displacement during the motion of the particle from O to P, we have

Initial velocity	: $u\ sin\ \alpha$	Formula	: $s = ut + \frac{1}{2}at^2$
Acceleration	: $-g$	$\therefore$ y= $u\ sin\ \alpha - \frac{1}{2}gt^2$. ......(2)	
Time	: t		
Distance travelled	: y		

Now (1) and (2) are the parametric equations of the trajectory. Its xy equation is obtained by eliminating the parameter t as

$$y = u\ sin\ \alpha \left\{\frac{x}{u\ cos\ \alpha}\right\} - \frac{g}{2}\left\{\frac{x}{u\ cos\ \alpha}\right\}^2$$

or $\qquad$ y= $x \tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$.

This is a second degree satisfying the relation "h^2 −ab=0". So it represents a parabola. Hence the trajectory is a parabola.

Second Method (Vector Method). Choose the x,y axes as in the first method and P(x, y) as the position of the particle (of mass m) at time t.

If $\bar{\imath}$ and $\bar{\jmath}$ are the unit vectors in the Ox and Oy directions, then the position vector of P is

$$\bar{r} = x\,\bar{\imath} + y\bar{\jmath}.$$

The only force acting on the particle is its own weight mg ($-\bar{\jmath}$). So the equation of motion,

m$\ddot{\bar{r}} = \bar{F}$, the particle is

$$m\ddot{\bar{r}} = -mg\,\bar{\jmath} \text{ or } \ddot{\bar{r}} = -g\,\bar{\jmath}.$$

Integration of this with respect to t,gives

$$\dot{\bar{r}} = -gt\,\bar{\jmath} + \bar{A},$$

where $\bar{A}$ is a constant vector. But, when t=0, $\dot{\bar{r}} = u\cos\alpha\,\bar{\imath} + u\sin\alpha\bar{\jmath}$, because the components of the initial velocity u in the $\bar{\imath}$ and $\bar{\jmath}$ directions are respectively $u\cos\alpha$ and $u\sin\alpha$. So,

$$u\cos\alpha\,\bar{\imath} + u\sin\alpha\bar{\jmath} = -g(0)\,\bar{\jmath} + \bar{A} \text{ or } \bar{A} = u\cos\alpha\,\bar{\imath} + u\sin\alpha\bar{\jmath}.$$

$$\dot{\bar{r}} = -gt\,\bar{\jmath} + (u\cos\alpha\,\bar{\imath} + u\sin\alpha\bar{\jmath})$$

$$= u\cos\alpha\,\bar{\imath} + (u\sin\alpha - gt)\bar{\jmath}$$

Once again integrating, we get

$$\bar{r} = u\cos\alpha t\,\bar{\imath} + (u\sin\alpha t - \tfrac{1}{2}gt^2)\bar{\jmath} + \bar{B}.$$

But, when t=0,$\bar{r} = \bar{0}$. So,

$$\bar{0} = u\cos\alpha(0)\,\bar{\imath} + \left\{(u\sin\alpha(0) - \tfrac{1}{2}g(0)^2)\right\}\bar{\jmath} + \bar{B}.$$

That is, $\bar{B} = \bar{0}$. Thus we get

$$\bar{r} = u\cos\alpha t\,\bar{\imath} + (u\sin\alpha t - 12gt2)\bar{\jmath}.$$

$\therefore \qquad x = u\, cos\alpha t$

$$y = u\, sin\alpha t - \frac{1}{2}gt^2.$$

Eliminating t from these two equations, we obtain the equation of the trajectory to be

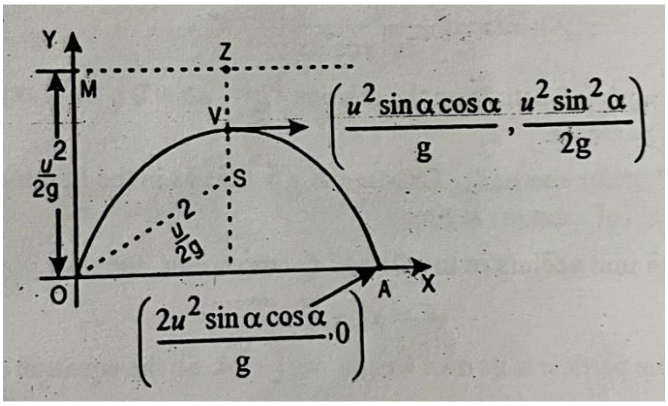
$$y = x\, tan\alpha - \frac{gx^2}{2u^2cos^2\alpha}$$

This is a seconddegree equation satisfying the relation"h^2 −ab=0". So it represents a parabola. Hence the trajectory is a parabola.

Remark. The above equation can be rewritten as

$$\left(x - \frac{u^2sin\alpha cos\alpha}{g}\right)^2 = \frac{2u^2cos^2\alpha}{g}\left(y - \frac{u^2sin^2\alpha}{2g}\right)$$

which shows that the parabola has a downward vertical as its axis



Its latus-rectum $(2u^2cos^2\alpha)$/g and its vertex v is

$$\left(\frac{u^2\, sin\alpha\, cos\alpha}{g}, \frac{u^2sin^2\alpha}{2g}\right)$$

Height of the directrix. Let Z be the point on the directrix vertically above V and M, the point on the directrix vertically above O.Then the height of the directrix above O is

OM= y coordinate of V+VZ

= y coordinate of V+$\frac{1}{4}$(latus rectum)

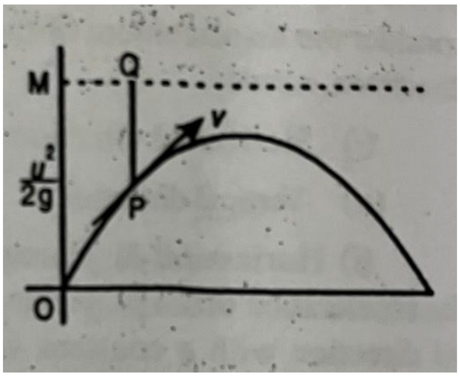
$$= \frac{u^2sin^2\alpha}{2g} + \frac{1}{4}\left(\frac{2u^2cos^2\alpha}{g}\right) = \frac{u^2}{2g}$$

Since $u^2/2g$is independent of the angle of projection α, we achieve that, *if several particles are projected from a point with the same speed u but in different directions (in the same vertical plane), their trajectories have the same horizontal line as their directrices which is at a height* $u^2/2g$*from the point of projection.*

Distance of focus from the point of projection. If S is the focus of the trajectory, then from the focus–directrix property of the parabola, that is, fromSO=OM, we get that OS=$u^2/2g$.From this we achieve that,

if several particles are projected from a point with the same speed u but in different directions (in the same vertical plane), then their trajectories have their foci on a circle whose centre is the point of projection and radius is $u^2/2g$.

Bookwork 4.2. The speed of a projectile at any point on its path equals the speed of a particle acquired by it in falling from the directrix to the point.



Let the projectile be projected from a point O with a speed u and with an angle of projection α.Let M be the point on the directrix of its path vertically above O. Then we know that the height of the directrix above the point of projection is

$$OM=\frac{u^2}{2g}$$

So also, if P is the position of the projectile at an arbitrarily chosen moment and if v is its speed then,

$$PQ=\frac{v^2}{2g}$$

where Q is the point on the directrix vertically above P because, when the projectile is moving at P with speed v, it can be assumed that the projectile is then projected, from there with a speed v along the tangent at P.

When the other particle falls through the distance QP, starting from rest at Q, its velocity at P, say V, is obtained, from the formula $v^2 = u^2 + 2as$, as

$$V^2 = 0 + 2g\,PQ = 2g\frac{v^2}{2g} = v^2 \text{ or } V=v.$$

This completes the proof.

Maximum height. From the equation of the trajectory we see that the maximum height attained by the particle is the y-coordinate of the vertex, namely,

$$\frac{u^2 sin^2\alpha}{2g}.$$

Horizontal range. If the horizontal range is R, then (R,0) is a point on the parabola and hence substituting in the equation of the parabola,

$$0 = R\tan\alpha - \frac{gR^2}{2u^2cos^2\alpha} \text{ or } R\left(\tan\alpha - \frac{gR}{2u^2cos^2\alpha}\right)=0.$$

R=0 corresponds to the initial position and the range is

$$R=\frac{2u^2cos^2\alpha}{g}.\tan\alpha = \frac{2u^2\sin\alpha\cos\alpha}{g}.$$

$\therefore$ $$R= \frac{u^2\sin 2\alpha}{g}.$$

We obtain such results differently in the following bookwork.

4.1.3 Results pertaining to the motion of a projectile.

Bookwork 4.3. A particle is projected from a point O on the ground with a velocity u inclined to the horizontal at an angle α. It hits the ground at A. To find

(i) Maximum height H attained by the particle

(ii) Time taken to attain the maximum height

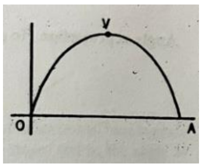
(iii) Time of flight (from O to A)

(iv) Horizontal range R (Range on the ground)

(v) Velocity at time t.

Choose the vertically upward direction as the positive direction to measure the distance of the particle.

(i) Maximum height H.



Let V be the topmost point of the trajectory. Then, for themotion from O to V, considering the vertical displacement, we have

(ii) **Time taken to reach the topmost point.** For the motion from O to V, considering the verticaldisplacement, we have

Initial velocity : u sinα	Formula. $v = u + at,$
Acceleration :$-g$	0=u sin$\alpha - gt$
Final velocity : 0	$\therefore t = \frac{\text{u sin}\alpha}{g}$
Time : H	

(iii) **Time of flight.** For the motion from O to A, considering the vertical displacement, we have

Initial velocity : u sinα	Formula. $s = ut + \frac{1}{2}at^2,$
Acceleration :$-g$	0=u sin$\alpha T - \frac{g}{2}T^2$
Final velocity : 0	$\therefore T = \frac{\text{2u sin}\alpha}{g}$ or 0.
Time : H	

T=0 corresponds to the initial position.

(iv) Horizontal range R.

Horizontal velocity (a constant) : u cosα

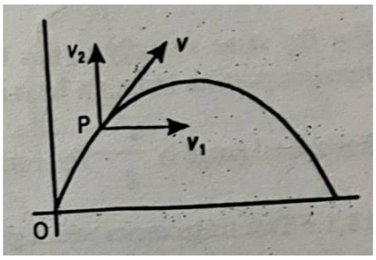
Time of flight T :$\frac{\text{2u sin}\alpha}{g}$

R= (Horl. Vel.)×(time) =u cos$\alpha \times \frac{\text{2u sin}\alpha}{g}$

$= \frac{u^2 sin2\alpha}{g}$

Note:R= $\frac{2}{g}$ (Initial horizontal velocity)×(Initial vertical velocity)

(iv) Velocity at time t.



Let the particle be at P at time t. Let its velocity there be V and its horizontal and upward vertical components be v_1,v_2. Since the horizontal components is always u cosα,

$$v_1 = \text{u } \cos\alpha$$

Considering the vertical displacement, for the motion from O to P, we have

Initial velocity: u sinα	Formula. $v = u + at$,
Acceleration:$-g$	$v_2 = \text{u } \sin\alpha - gt$.
Final velocity :t	
Time :v_2	

∴ $$V=\sqrt{v_1^2 + v_2^2} = \sqrt{(\text{u } \cos\alpha)^2 + (\text{u } \sin\alpha - gt)^2}$$

$$= \sqrt{u^2 - 2u\ g\ sin\alpha t + g^2t^2}.$$

The angle made by this velocity with the horizontal is

$$tan^{-1}\frac{v_2}{v_1} = tan^{-1}\frac{\text{u } \sin\alpha - gt}{\text{u } \cos\alpha}$$

Bookwork 4.4. To verify, in the case of a projectile,

K.E+P.E= a constant

Let m the mass of the projectile. Then, with usual meanings, at time t,

$$\text{K.E} = \frac{1}{2}mv^2 = \frac{1}{2}m(u^2 - 2u\ g\ sin\alpha t + g^2t^2). \quad \ldots\ldots(1)$$

At time t,let the vertical height of the particle be h. Then, considering the vertical displacement, we have

Initial velocity: u sinα	
Acceleration:$-g$	$\therefore h = usin\alpha\, t - \frac{1}{2}gt^2$
Final velocity :t	
Time :h	

Thus, choosing the point of projection O as the standard point, P.E. at time t,

$$\text{P.E}= mg \times h = mg(usin\alpha\, t - \tfrac{1}{2}gt^2) \qquad \ldots\ldots(2)$$

$$\text{K.E+P.E}= \tfrac{1}{2}mu^2$$

which is a constant.

4.1.4 Maximum horizontal range for a given velocity.

Suppose a particle is projected with a velocity u and with the angle of projection α. Then we know that its horizontal range is

$$\frac{u^2 sin2\alpha}{g}.$$

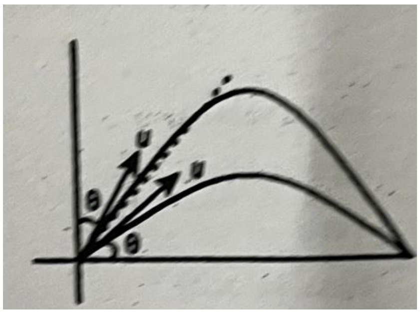
For the same velocity u, this range assumes different values for different α's and it attains the maximum value when $sin2\alpha$ is a maximum. But the maximum value of $sin2\alpha$ is 1. So the maximum horizontal range is$\frac{u^2}{g}$.The corresponding angle of projection is given by2α=90^0. So it is 45^0.

4.1.5 Two trajectories with a given speed and range

If the given speed is u and the given horizontal range is R, (R<$\frac{u^2}{g}$), then the corresponding angle of projection α is given by

$$\frac{u^2 sin2\alpha}{g} = R$$

or $$sin2\alpha = \frac{gR}{u^2} \qquad \ldots\ldots(1)$$

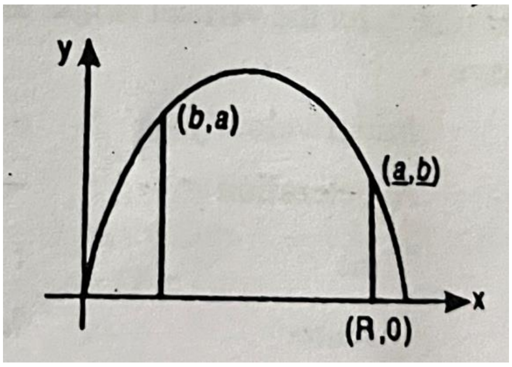


So, from trigonometry, we see that there are two values for 2α satisfying (1). The sum of these values is 180^0. In other words, there are two values for α whose sum is 90^0.Hence, for a given speed, there are two trajectories giving the same horizontal range. If the two values of α are θ and $90^0-\theta$, it can be seen that the directions corresponding to these two values are equally inclined to the direction which gives the maximum horizontal range because

$$(45^0)-(\theta) = (90^0-\theta)-(45^0).$$

EXAMPLES

Example 1. A ball is projected so as to just clear two parallel walls, the first of height a at a distance b from the point of projection and the second of height$b$at a distance a from the point of projection. Supposing the path of the ball to lie in a plane perpendicular to the walls, find the range on the horizontal plane and show that the angle of projection exceeds $tan^{-1}3$.



With the usual notations, the equation of the trajectory is

$$y= x\tan\alpha - \frac{gx^2}{2u^2cos^2\alpha}$$

Range. The trajectory passes through the points

(b,a), (a, b), (R,0),

where R is the horizontal range. So,

$$a = b\tan\alpha - \frac{gb^2}{2u^2\cos^2\alpha} \qquad \text{.........(1)}$$

$$b = a\tan\alpha - \frac{ga^2}{2u^2\cos^2\alpha} \qquad \text{.........(2)}$$

$$0 = R\tan\alpha - \frac{gR^2}{2u^2\cos^2\alpha} \qquad \text{.........(3)}$$

Now, eliminating $tan\ \alpha$ and $\frac{g}{2u^2cos^2\alpha}$ from (1), (2), (3), we get the eliminant as

$$\begin{vmatrix} a & b & b^2 \\ b & a & a^2 \\ 0 & R & R^2 \end{vmatrix} = 0 \ or \begin{vmatrix} a & b & b^2 \\ b & a & a^2 \\ 0 & 1 & R \end{vmatrix} = 0$$

i.e.,$a(aR - a^2) - b(bR) + b^2(b) = 0 \quad or \quad k(a^2 - b^2) = a^3 - b^3$

$\therefore \qquad R= \frac{a^3-b^3}{a^2-b^2} = \frac{a^2+ab+b^2}{a+b}.$

Angle of projection. To get tanα consider $a^2 \times (1) - b^2 \times (2)$. Then

$$(ba^2 - ab^2)tan\alpha = a^3 - b^3.$$

$\therefore \qquad \tan\alpha = \frac{a^3-b^3}{ab(a-b)} = \frac{a^2+ab+b^2}{ab}.$

$\therefore \qquad \alpha = tan^{-1}\frac{a^2+ab+b^2}{ab}.$

Restriction on $\boldsymbol{\alpha}$. The angle α is such that

$$tan\alpha = \frac{a^2+ab+b^2}{ab} = \frac{(a-b)^2+3ab}{ab}$$

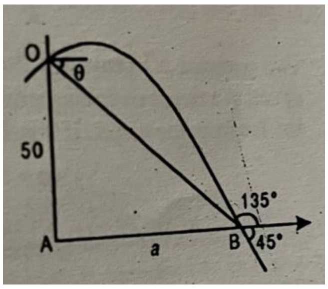
$$= \frac{(a-b)^2}{ab} + 3.$$

But $\frac{(a-b)^2}{ab}$ is positive and 0 when $a = b$.

$\therefore \qquad \tan\alpha > 3 \ \ or\ \alpha > tan^{-1}3.$

Example 2. A particle projected from the top O of a wall AO, 50 m. high, at an angle of 30^0 above the horizon, strikes the level ground through A at B at an angle of 45^0. Show that the angle of depression of B from O is

$tan^{-1}\frac{\sqrt{3}-1}{2\sqrt{3}}$.



With O as the origin, etc., the equation of the trajectory is

$$y= x\ tan\ \alpha - \frac{gx^2}{2u^2cos^2\alpha} \qquad \text{...........(1)}$$

If AB$=\ a$, then B is

(a,$-$50)

which satisfy (1). So

$$-50 = a\ tan\ \alpha - \frac{ga^2}{2u^2cos^2\alpha} \qquad \text{............(2)}$$

But the slope of the curve at A is $tan\ 135^0$ or -1.Finding the slope by differentiation,

$$\left(\frac{dy}{dx}\right)_{(a,-50)} = \left[tan\ \alpha\ - \frac{gx}{u^2cos^2\alpha}\right]_{x=a} = tan\ \alpha\ - \frac{ga}{u^2cos^2\alpha}$$

$$\therefore \qquad -1 = tan\ \alpha\ - \frac{ga}{u^2cos^2\alpha}. \qquad \text{......... (3)}$$

Thus (2) $-\frac{a}{2}\times$ (3) and $\alpha = 30^0$ give

$$-50 = a\left(-\frac{1}{2}+\frac{1}{2}tan\alpha\right);\ \ a = \frac{100\sqrt{3}}{\sqrt{3}-1}.$$

If θ is the angle of depression at O,

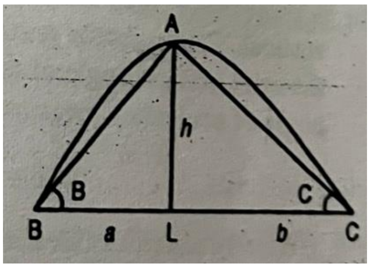
$$tan\theta = \frac{OA}{OB} = 50\frac{\sqrt{3}-1}{100\sqrt{3}} = \frac{\sqrt{3}-1}{2\sqrt{3}}.$$

Example 3. (i) A particle is projected over a triangle from one end of its horizontal base to graze the vertex and fall at the other end of the base. If B and C are the base angles and α, the angle of projection, show that $tan\alpha = tanB + tanC$.

(ii) A particle is projected from a point P at an angle of 30^0 to the horizontal. If PQ is its horizontal range and if angles of elevation of the particle at P and Q at any instant of its flight are A and B respectively, show that$tanA + tanB = \frac{1}{\sqrt{3}}$.

(i) Let the projectile be projected from B with a velocity u. Let AL be the altitude through A and

$$AL = h, BL = a, LC = b.$$



If B is the origin and BC, the x axis, then A and C are

$$(a, h), (a + b, 0)$$

and the equation of the trajectory is

$$y = x\ tan\ \alpha - \frac{gx^2}{2u^2cos^2\alpha}.$$

This is satisfied by A and C. Therefore

$$h = a\ tan\ \alpha - \frac{ga^2}{2u^2cos^2\alpha},\quad 0 = tan\ \alpha - \frac{g(a+b)}{2u^2cos^2\alpha}.$$

Eliminating u, by multiplying the first by $a + b$ and the second by a^2 and then subtracting,

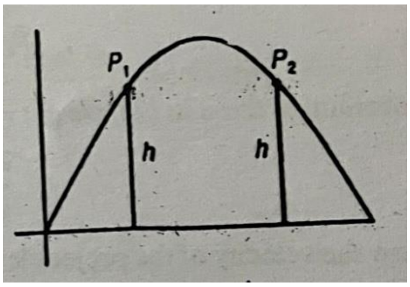
$$(a + b)h = ab\ tan\alpha.$$

$\therefore$ $$tan\alpha = \frac{h}{a} + \frac{h}{b} = tanB + tanC.$$

(ii) $tan\alpha = tanA + tanB \Rightarrow tan\ 30^0 = tanA + tanB.$

Example 4. A particle is projected with a velocity whose horizontal and vertical components are U and V. If P_1 and P_2 are the points on its trajectory at the same height h above the point of projection, show that

$$P_1P_2 = \frac{2U}{g}\sqrt{v^2 - 2gh}\,.$$



With usual notations, we have

$$U = u\cos\alpha$$

$$V = u\sin\alpha$$

Let the particle attain a height h at time t. Then

$$h = usin\alpha\, t - \frac{1}{2}gt^2$$

or $\quad h = V\,t - \frac{1}{2}gt^2$

or $$gt^2 - 2Vt + 2h = 0.$$

The roots of this quadratic are the two values of t at which the particle is at the height h. Let them be t_1, t_2. Then

$$t_1 + t_2 = \frac{2V}{g},\ t_1t_2 = \frac{2h}{g}$$

$$\therefore \quad (t_2 - t_1)^2 = (t_1 + t_2)^2 - 4t_1t_2 = \frac{4V^2}{g^2} - \frac{8h}{g} = 4\left(\frac{v^2 - 2gh}{g^2}\right)$$

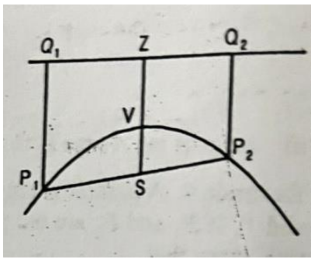
$\therefore \quad t_2 - t_1 = \frac{2}{g}\sqrt{v^2 - 2gh}.$

So the time interval for the particle to move from P_1 to P_2 is $t_2 - t_1$.

$\therefore \quad P_1P_2 = (t_2 - t_1)U = \frac{2U}{g}\sqrt{v^2 - 2gh}$

Example 5. If v_1 and v_2 are the velocities of a projectile at the ends of a focal chord of its path and v, the horizontal component of its velocity, show that

$$\frac{1}{v_1^2} + \frac{1}{v_2^2} = \frac{1}{v^2}.$$



Let P_1SP_2 be the focal chord and l, the semilatus rectum of the parabola. Then we know, from geometry, that

$$\frac{1}{SP_1}+\frac{1}{SP_2}=\frac{2}{l}. \qquad \text{………….. (1)}$$

If Q_1, Z, Q_2 are the points on the directrix vertically above P_1, V, P_2. Then

$$SP_1 = P_1Q_1,\ SP_2 = P_2Q_2,\ \frac{l}{2} = VZ$$

Thus, (1) becomes

$$\frac{1}{P_1Q_1}+\frac{1}{P_2Q_2}=\frac{1}{VZ} \qquad \text{………… (2)}$$

But we know that the velocity of a projectile at any point on its path is the velocity of a particle falling freely from the directrix to that point.

Thus,

$$v_1^2 = 2g\,P_1Q_1,\ v_2^2 = 2g\,P_2Q_2,\ v^2 = 2g\,VZ$$

or $P_1Q_1 = \frac{v_1^2}{2g},\ P_2Q_2 = \frac{v_2^2}{2g},\ VZ = \frac{v^2}{2g}$

Substituting these in (2), we get

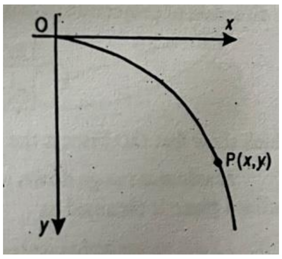
$$\frac{2g}{v_1^2}+\frac{2g}{v_2^2}=\frac{2g}{v^2} \text{ or } \frac{1}{v_1^2}+\frac{1}{v_2^2}=\frac{1}{v^2}.$$

Now the velocity of the projectile at V, namely v, is the horizontal component of its velocity.

4.1.6 Projectile projected horizontally.

Bookwork 4.5. A particle is projected horizontally from a height. To show that the equation of its path can be put in the form

$$x^2 = \frac{2u^2}{g}y.$$



Choose the horizontal line and the vertical downward line through the point of projection, say O, as x, y axes. Let P(x, y) be the position of the particle at time t. Now the horizontal component of the velocity is the constant u.

$\therefore$ $x = ut.$ ………….. (1)

Choosing the y direction as the positive direction,

Initial vel. : 0	
Acceleration: g	$\therefore y = \frac{1}{2}gt^2$ ………..(2)
Time :t	
Distance :y	

Eliminating t from (1) and (2), we get the equation of the trajectory as

$$x^2 = \frac{2u^2}{g}y.$$

Bookwork 4.6.A is a point on the ground. O is a point above A such that AO = h. A particle projected horizontally from O hits the ground at B. To find the time of flight T and the range AB.

Choosing the downward direction as the positive direction to measure the distance in the vertical displacement, we have, for the entire motion,

Initial velocity : 0	Formula. $s = ut + \frac{1}{2}at^2$,
Acceleration :g	$h = \frac{1}{2}gT^2$
Distance :h	$T = \sqrt{\frac{2h}{g}}.$
Time :T	

The horizontal component of the velocity is always u. Therefore

$$AB = (Horizontal\ velocity) \times (Time) = u\sqrt{\frac{2h}{g}}.$$

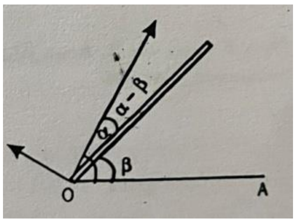
4.2 Projectile projected on an inclined plane

In this section we obtain the time of flight and range of a projectile projected to travel above a line of greatest slope of an inclined plane.

Bookwork 4.7. When a particle is projected from a point O on a plane of inclination β with a velocity u making an angle α with the horizontal, to find

(i) T, the time of flight

(ii) R, the range on the plane.



Motion perpendicular to the plane. Let us consider the direction perpendicular to the plane in the upward sense. The angle between this direction and the initial velocity is

$$90^0 - (\alpha - \beta).$$

So the component of the initial velocity in this direction is

$$u \sin(\alpha - \beta).$$

Now earth's gravity is the only force acting on the particle. So the acceleration of the particle is g vertically downward. Hence the acceleration of the particle in the upward perpendicular direction is

$$-g\ cos\beta.$$

Time of flight T. Let the particle hit the inclined plane at B. Then, for the motion from O to B, in the upward perpendicular direction, we have

Initial velocity :usin$(\alpha - \beta)$.

Acceleration : $-g\ cos\beta$.

Time :T

Distance :0(At B, the distance perpendicular to the plane is zero.)

Therefore, from $s = ut + \frac{1}{2}at^2$,we get

$$0 = \text{u}\sin(\alpha - \beta)T - \frac{1}{2}\ g\ cos\beta T^2 .\text{So} T = \frac{2\text{u}\sin(\alpha-\beta)}{g\ cos\beta}$$

Range on the inclined plane. Let N be the foot of the perpendicular from B to the horizontal line through O. Then ON is the horizontal displacement of the projectile when it moves from O to B. So

$$\text{ON} = (Horizontal\ velocity) \times T$$

$$= (ucos\alpha)\frac{2\text{usin}(\alpha-\beta)}{g\ cos\beta}$$

$$= \frac{2u^2\text{cos}\alpha\text{sin}(\alpha-\beta)}{g\ cos\beta}.$$

Now, from $\Delta ONB, cos\beta = \frac{ON}{OB} = \frac{ON}{R}\ \ and\ so$

$$R = \frac{ON}{cos\beta} = \frac{2u^2 cos\alpha sin(\alpha-\beta)}{g\ cos^2\beta}.$$

An important note. If$\alpha > 90^0$ then the algebraic value of the range is negative, that is, the actual range is down the plane.

Alternative method. The range R on the inclined plane may also be obtained from the equation of the trajectory

$$\text{y} = x\ tan\ \alpha - \frac{gx^2}{2u^2cos^2\alpha}.$$

Now B is $(R\ cos\ \beta, R\ sin\ \beta)$ and it lies on the trajectory. So it satisfies the equation. Therefore, substituting, we get

$$R\ sin\ \beta = R\ cos\ \beta\ tan\alpha\ - \frac{gR^2cos^2\beta}{2u^2cos^2\alpha}$$

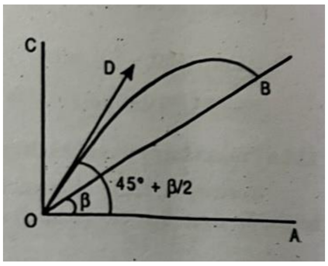
Division by R gives

$$\text{R}\frac{gcos^2\beta}{2u^2cos^2\alpha} = cos\ \beta\ tan\alpha - \ sin\ \beta = cos\ \beta\frac{sin\ \beta}{cos\ \beta} - sin\ \beta$$

$$= \frac{\cos\beta \sin\alpha - \sin\beta \cos\alpha}{\cos\alpha} = \frac{\sin(\alpha-\beta)}{\cos\alpha}$$

$\therefore$ $$R = \frac{2u^2 \sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

4.2.1 Maximum range on an inclined plane



Since $2\sin(\alpha-\beta)\cos\alpha = \sin(2\alpha-\beta) - \sin\beta,$ the range R on the inclined plane can be rewritten as

$$R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta) - \sin\beta].$$

Hereβ is a constant. So, for a given speed u, R is a maximum when$\sin(2\alpha-\beta)$ is a maximum, that is, when $\sin(2\alpha-\beta) = 1$. So the maximum R for a given speed u, is

$$\frac{u^2}{g\cos^2\beta}(1 - \sin\beta) = \frac{u^2(1-\sin\beta)}{g(1-\sin^2\beta)} = \frac{u^2}{g(1+\sin\beta)} \qquad \text{.........(1)}$$

The value of α corresponding to the maximum range on the inclined plane is obtained from

$$2\alpha - \beta = 90^0$$

as

$$\alpha = 45^0 + \frac{1}{2}\beta.$$

If OA is the horizontal line, OC the vertical line through O and OD the direction of projection giving the maximum range, then

$$\angle BOD = (45^0 + \frac{1}{2}\beta) - \beta = 45^0 - \frac{1}{2}\beta,$$

$$\angle COD = 90^0 - (45^0 + \frac{1}{2}\beta) = 45^0 - \frac{1}{2}\beta$$

which show that OD bisects the angle between the inclined plane and the vertical line.

Maximum range down an inclined plane. For a given speed u, the maximum range down the inclined plane is obtained as

$$\frac{u^2}{g(1-\sin\beta)} \qquad \ldots\ldots\ldots (1)$$

by replacing β with $180^0+\beta$ in (1).

EXAMPLES

Example 1.A particle projected with a speed u strikes at right angles a plane through the point of projection, inclined at an angle β to the horizon. If α, T and R are the angle of projection, the time of flight and the range on the inclined plane, show that

(i) $cot\,\beta = 2tan(\alpha-\beta)$

(ii) $cot\,\beta = tan\,\alpha - 2tan\beta.$

(iii) $T = \frac{2u}{g\sqrt{1+3sin^2\beta}}$

(iv) $R = \frac{2u^2 sin\beta}{g(1+3sin^2\beta)}$

(i) Condition for perpendicular hit. See the Bookwork. Let OB be the range. Since the particle hits the plane perpendicularly at B, the velocity component along the plane at B is zero, that is, at $t = T$ it is zero. So, from $v = u + at$,

$$0 = \text{u}\cos(\alpha-\beta) - \ g\,sin\beta T \qquad \ldots\ldots\ldots (1)$$

But, in general,

$$T = \frac{2\text{u}\,sin(\alpha-\beta)}{g\,cos\beta} \qquad \ldots\ldots\ldots (2)$$

Thus the condition for perpendicular hit is

$$0 = \text{u}\sin(\alpha-\beta) - \frac{1}{2}\,g\,cos\beta\,\frac{2\text{u}\,sin(\alpha-\beta)}{g\,cos\beta}$$

or

$$\cot\beta = 2\,tan(\alpha-\beta) \qquad \ldots\ldots\ldots (3)$$

(ii) Expansion of$tan(\alpha-\beta)$ yields the result

$\cot\beta = \ tan\alpha - 2tan\beta.$

(iii) Finding$u\ sin(\alpha-\beta), u\ \ cos(\alpha-\beta)$ from (2), (1) and squaring and adding, we get

$$u^2 = (sin^2\beta + \frac{1}{4} cos^2\beta)g^2T^2.$$

$\therefore$ $$g^2(4sin^2\beta + cos^2\beta)T^2 = 4u^2 \quad or \quad g^2(3sin^2\beta + 1)T^2 = 4u^2.$$

$\therefore$ $$T = \frac{2u}{g\sqrt{1+3sin^2\beta}} \qquad \text{……... (4)}$$

(iv) In general, considering the displacement along and up the plane in the motion from O to B, we have

Initial velocity :u cos $(\alpha - \beta)$ | Time: T

Acceleration :$-g\ sin\beta$Distance : R.

From "$s = ut + \frac{1}{2}at^2$", we get

$$R = \text{u sin}\,(\alpha - \beta)T - \frac{1}{2}\ g\ cos\beta T^2 \qquad \text{……… (5)}$$

(5)$-T \times$ (1) gives

$$R = T^2(-\frac{1}{2}\ g\ sin\beta + g\ sin\beta) = T^2\left(\frac{1}{2}\ g\ sin\beta\right)$$

$$= \frac{4u^2}{g^2(1+3sin^2\beta)} \cdot \frac{1}{2} gsin\beta \quad by(4)$$

$$= \frac{2u^2sin\beta}{g(1+3sin^2\beta)}.$$

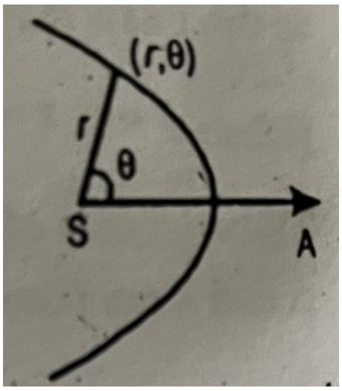
UNIT V

CENTRAL ORBITS

5.1 GENERAL ORBITS

In this section we get the orbits of particles when their velocity components or their acceleration components are given. Central orbit is an orbit under a central force.

Central force:



When a particle is subject to the action of a force which is always either towards or away from a fixed point. If the particle is said to be under the action of a central force.

The velocity and acceleration components in the radial and transverse directions are as follows.

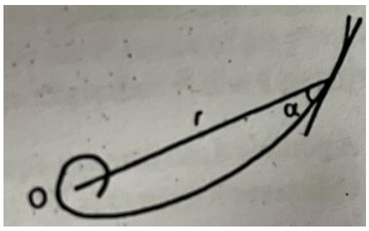
Velocity components: $\dot{r}$, r $\dot{\theta}$

Acceleration components: $\ddot{r} - r\dot{\theta}^2, \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$,

Conic: Now it is necessary to remember that the polar equation of a conic is

$$\frac{l}{r} = 1 + ecos\theta,$$

if one focus S is the pole and SA is the initial line as in the figure. l is the semilatus rectum.



Equiangular spiral:

Equiangular spiral is a curve which is such that the angle between the radius vector and the respective tangent is a constant, say α. Its polar equation is

$$r = A\, e^{(cot\alpha)\theta},$$

where A is a constant.

EXAMPLES

Example 1 : The velocities of a particle along and perpendicular to the radius vector are λrand $\mu\theta$. Find the path and show that the acceleration components along and perpendicular to the radius vector are $\lambda^2 r - \frac{\mu^2\theta^2}{r}$, $\mu\theta\left(\lambda + \frac{\mu}{r}\right)$.

Solution:

It is given that: $\dot{r} = \lambda r$ or $\frac{dr}{dt} = \lambda r, r\,\dot{\theta} = \mu\theta$ or $\frac{d\theta}{dt} = \mu\theta$.

Dividing (1) by (2), we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\lambda r}{\mu\theta} or \frac{1}{r^2}dr = \frac{\lambda}{\mu}\frac{1}{\theta}d\theta.$$

On integration, we get the equation of the path as

$$-\frac{1}{r} = \frac{\lambda}{\mu}\ log\theta + A.$$

The acceleration along the radius vector is

$$\ddot{r} - r\dot{\theta}^2 = \frac{d\dot{r}}{dt} - r\dot{\theta}^2 = \frac{d(\lambda r)}{dt} - r\left(\frac{\mu\theta}{r}\right)^2 \quad \text{by (1), (2)}$$

$$= \lambda\frac{dr}{dt} - \frac{\mu^2\theta^2}{r} = \lambda(\lambda r) - \frac{\mu^2\theta^2}{r}$$

$$= \lambda^2 r - \frac{\mu^2\theta^2}{r}.$$

The acceleration perpendicular to the radius vector is

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{\mu\theta}{r}\right) \quad \text{by (2)}$$

$$= \frac{\mu}{r}\left[\dot{r}\theta + r\dot{\theta}\right] = \frac{\mu}{r}[\lambda r\theta + \mu\theta]$$

$$= \mu\theta\left[\lambda + \frac{\mu}{r}\right].$$

Example 2 : The velocities of a particle along and perpendicular to the radius vector from a fixed origin are a and b. Find the path and the acceleration along and perpendicular to the radius vector.

Solution:

It is given that $r = a, r\dot{\theta} = b$.

Dividing the first by the second,

$$\frac{\dot{r}}{r\dot{\theta}} = \frac{a}{b}. \text{ So} \frac{1}{r}\frac{dr}{d\theta} = \frac{a}{b}$$

$$\therefore \frac{1}{r}dr = \frac{a}{b}d\theta.$$

Integration gives the equation of the path as

$$\log r = \frac{a}{b}\theta + log C or \frac{r}{C} = e^{a/_b\theta} or r = Ce^{a/_b\theta}$$

which is the equation of an equiangular spiral. Also $\dot{r} = 0$ and $\dot{\theta} = {}^{b}/_{r}$. So the acceleration components are

(i) $\ddot{r} - r\dot{\theta}^2 = 0 - r\frac{b^2}{r^2} = -\frac{b^2}{r}$

(ii) $\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{1}{r}\frac{d}{dt}(br) = \frac{b\dot{r}}{r} = \frac{ab}{r}$.

Example 3 : Show that the path of a point P whose velocity is such that its components in a fixed direction and in the direction perpendicular to the line joining P into a fixed point O are respectively the constants u and v, is a conic with a focus at O and eccentricity $\frac{u}{v}$.

Solution:

Let O be the pole and OA, the given fixed direction, the initial line. Then the components of the velocity u in the radial and transverse directions are

$$u\, cos\theta, -u\, sin\theta,$$

$$\therefore \dot{r} = ucos\theta, r\,\dot{\theta} = v - usin\theta,$$

Dividing $\dot{r}$ by $r\dot{\theta}, we\ get$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{ucos\theta}{v - usin\theta},$$

Integration gives

$$\log r = -\log(v - usin\theta) + \log C$$

$$r = \frac{C}{v - usin\theta} \quad or \quad \frac{C}{r} = v - usin\theta$$

$$\frac{C/_v}{r} = 1 - \frac{u}{v}sin\theta\ .$$

So, the path is a conic with its focus at O and eccentricity is $\frac{u}{v}$.

Example 4 : A point P describes an equiangular spiral with a constant angular velocity about the pole O.Show that its acceleration varies as OP and is in a direction making with the tangent at Pthe same constant angle that OP makes.

Solution:

The equation of an equiangular spiral is$r = a\, e^{0cot\alpha}$, where a, α are constants.α is the angle between the tangent at any point and the radius vector to that point.Let the constant angular velocity be ω.Now,

$$\dot{r} = ae^{0cot\alpha}cot\alpha\dot{\theta} = \omega cot\alpha r\ ,$$

$$\ddot{r} = \omega cot\alpha\dot{r} = \omega^2 cot^2\alpha r\ .$$

Then the radial and the transverse components of acceleration are

$$\ddot{r} - r\dot{\theta}^2 = \omega^2 \cot^2 \alpha\, r - r\omega^2 = \omega^2 r(\cot^2 \alpha - 1)$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{\omega}{r}(2r\dot{r}) = \frac{\omega}{r}(2r\omega cot\alpha r) = 2\omega^2 rcot\alpha.$$

∴ Acceleration $=\sqrt{\omega^4\, r^2\, (\cot^2 \alpha\ -\ 1)^2\ +\ 4\, \omega^4\, r^2 \cot^2 \alpha}$

$$= \sqrt{\omega^4 r^2 (1\ +\ \cot^2\alpha)^2}$$

$$=\ \omega^2\, r\, \mathrm{cosec}^2\, \alpha,$$

which is proportional to r.

If β is the angle between the radial direction and the direction of the acceleration, then

$$tan\beta = \frac{\text{Transverse component}}{\text{Radial component}} = \frac{2\,\omega^2\, r \cot\alpha}{\omega^2\, r\, (\cot^2\alpha\ -\ 1)}$$

$$= \frac{2\tan\alpha}{1\ -\ \tan^2\alpha} = tan2\alpha.$$

So $\beta = 2\alpha$. But the angle between the radial direction and the tangent is α.So, the anglebetween the tangent and the acceleration is α.

5.2 CENTRAL ORBITS

In this chapter we study the motion of a particle subject to the action of a central force.

Central force: When a particle is subject to the action of a force which is always either towards or away from a fixed point, the particle is said to be under the action of a central force. That is, a central force is a forcewhose line of action always passes through a fixed point.

Centre of force:A central force is a force whose line of action always passes through a fixed point.The fixed point is called the centre of force.

Polar coordinates: To study the central motion of a particle we use polar coordinates (r, θ),choosing the centre of force as the pole and a fixed line through it as the initial line.

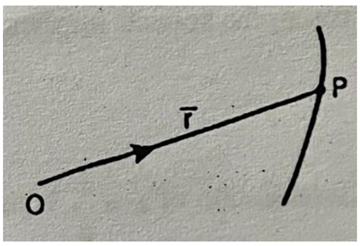
Notation: We shall denote the central force per unit mass by

$$\varphi(r)\,\hat{r},$$

where $\hat{r}$ is the unit vector in the radial direction in the sense in which r increasesand $\varphi(r)$ is a function of distance r of the particle from the centre of force.

Central orbit: The path described by a particle under a central force is called a central orbit.

Bookwork 5.1. To show that a central orbit is a plane curve.



Let us make the following assumptions:

O : Centre of force

P : Position of the particle at time t

$\vec{r}$: $\overline{OP}(r = OP)$

$\hat{r}$: Unit vector along OP

m : Mass of the particle

$\varphi(r)\,\hat{r}$: Central force per unit mass

Then the equation of motion of the particle is

$$m\,\ddot{r} = m\,\varphi(r)\,\hat{r} \quad \text{or} \quad \ddot{r} = \varphi(r)\,\hat{r} \qquad \ldots\ldots\ldots\ldots(1)$$

Let us consider, in particular,

$$\frac{d}{dt}(\bar{r} \times \dot{\bar{r}}) = \dot{\bar{r}} \times \dot{\bar{r}} + \bar{r} \times \ddot{\bar{r}}$$

$$= 0 + \bar{r} \times \varphi(r)\,\dot{\bar{r}}$$

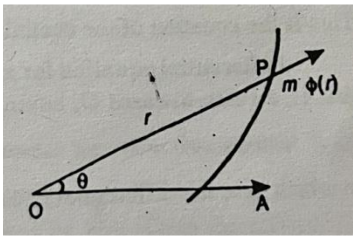
= 0.

This implies that $\bar{r} \times \dot{\bar{r}}$ is a constant vector, say $\vec{c}$. Then $\bar{r}$ or $\overline{OP}$ is always perpendicular to $\vec{c}$. So P is always in the plane through O and perpendicular to $\vec{c}$. Hence the motion of P is coplanar and the orbit is a plane curve.

5.2.1 Differential equation of a central orbit.

We now obtain the equation of a central orbit in polar coordinates.

Bookwork 5.2. To obtain the differential equation of a central orbit in polar coordinates.



Let us make the following assumptions:

O : Centre of force

O : Pole

OA : Initial line

P(r, θ) : Position of the particle at time t φ(r) : Central force per unit mass in the direction OP

m : Mass of the particle

The motion is a coplanar motion. We shall consider the equations of motion corresponding to the radial and transverse directions. We know that, in these directions, the acceleration components are

$$\ddot{r} - r\theta^2, \frac{1}{r}\frac{\dot{d}}{dt}(r^2\dot{\theta}).$$

The force components in these directions are

$$m\varphi(r), 0.$$

Therefore, the respective equations of motions are

$$m(\ddot{r} - r\dot{\theta}^2) = m\varphi(r) \; or \; \ddot{r} - r\dot{\theta}^2 = \varphi(r), \qquad \ldots\ldots. (1)$$

$$m\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \;\; or \;\; \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \qquad \ldots\ldots. (2)$$

Of these, the second implies that $r^2\dot{\theta}$ is a constant, say h. If $\frac{1}{r}$ is denoted by u, then

$$\dot{\theta} = \frac{h}{r^2} = hu^2. \qquad \ldots\ldots.. (3)$$

Differentiation of $r = \frac{1}{u}$ with respect to t gives

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}(hu^2) = -h\frac{du}{d\theta} \text{ by (3).} \qquad \ldots\ldots.. (4)$$

$$\therefore \frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}(hu^2) = -h^2u^2\frac{d^2u}{d\theta^2} \quad \text{by (3).} \qquad \ldots\ldots.. (5)$$

Substituting these values of $\ddot{r}$ and $\dot{\theta}$ in (1),

$$-h^2u^2\frac{d^2u}{d\theta^2} - r(hu^2)^2 = \varphi(r)$$

Since $r = {}^1/_u$, this can be written as

$$\frac{d^2u}{d\theta^2}+u=-\frac{\varphi(r)}{h^2u^2}.$$

This is the equation of the central orbit in polar coordinates.

Differential equation for an attractive central force: If the central force is an attractive one, that is, a force towards O, having a magnitude F per unit mass, then

$$\varphi(r)=-F$$

in which case the differential equation of the orbit becomes

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}.$$

Apse: If O is the pole and P is a point on a curve such that OP is perpendicular to the tangent at P, then P is an apse. If P is an apse OP is a maximum or minimum od r, Forexample, in an ellipse, if S is the pole, then the ends of the major axes are apses.

Maximum and minimum angular velocity: Since$r^2\,\dot{\theta}$is a constant in a central orbit, the angular velocity of the particle about O is a maximum or minimum according as the radius vector r is a minimum or maximum. So, at an apse, the angular velocity of the particle is either a maximum or minimum. The apses are given by$\frac{du}{d\theta}=0$.

Force per unit mass: The force per unit mass is

$$\varphi(r)\hat{r}=-h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)\hat{r}. \qquad \ldots\ldots. (6)$$

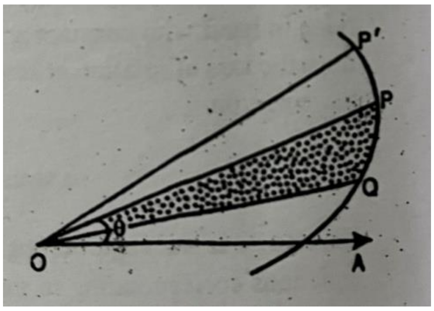
Velocity v at P: We know that in a coplanar motion the velocity components along the radialand transverse directions are

$$\dot{r}, r\,\dot{\theta}.$$

$$\therefore v^2=(\dot{r})^2+\left(r\dot{\theta}\right)^2=h^2\left(\frac{du}{d\theta}\right)^2+\frac{1}{u^2}(hu^2)^2 \qquad by\ (4),(3).$$

$$=h^2\left(\frac{du}{d\theta}\right)^2+h^2u^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]. \qquad \ldots\ldots..(7)$$

Areal velocity:



Let the initial position of the particle be Q and the position at time tbeP(r, θ).Let the area OQP swept by the radius vector moving from OQ to OP be A.Then

$$\frac{dA}{dt}$$

Is called the areal velocity of the particle.

Let p' be the position of the particle at time $t + \Delta t$, then

$$\Delta A = Area\ POP' = \frac{1}{2} OP.OP'.sin\Delta\theta$$

$$= \tfrac{1}{2} r(r + \Delta r)\Delta\theta \text{ Since } \Delta\theta \text{ is small.}$$

$$\therefore \frac{\Delta A}{\Delta t} = \frac{1}{2} r(r + \Delta r)\frac{\Delta\theta}{\Delta t}.$$

So the areal velocity of P is

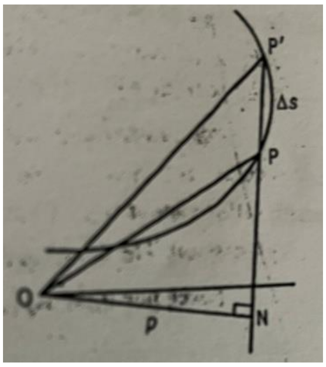
$$\Delta t \lim_{\rightarrow} 0 \frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}.$$

Constancy of the areal velocity in central orbit: In a central orbit $r^2\dot{\theta}$ is a constant which we have denoted by h. So,in a central orbit the areal velocity

$$\frac{1}{2} r^2 \dot{\theta}$$

is constant$\frac{h}{2}$.

Alternative form for areal velocity:



If $ON(=P)$is the perpendicular drawn from O to the chord $PP'$and the length of the arc PP' is Δs, then

$$\Delta A = \frac{1}{2} PP' \times ON \approx \frac{1}{2} \Delta s \times ON$$

$$\therefore \frac{\Delta A}{\Delta t} = \frac{1}{2} \frac{\Delta s}{\Delta t} \times ON$$

$$\therefore \frac{dA}{dt} = \frac{1}{2} \frac{ds}{dt} p = \frac{1}{2} vp = \frac{1}{2} pv,$$

where v is the velocity of the particle at P.

Remark: Now$\frac{1}{2} pv = \frac{h}{2}$ or$pv = h$. So the velocity at any point P is inversely proportional to the perpendicular drawn from O to the tangent at P.

Constancy of moment of momentum: The momentum of the particle is mv which is along the tangent.Its moment about O is

$$ON \times (mv) = m(pv) = mh.$$

Thus, the moment of momentum about O, otherwise known as angular momentum about O,is the constant mh.

Bookwork 5.3: To obtain the differential equation of a central orbit in $p - r$coordinates.

For any point P on the orbit, the radius vector$r(= OP)$ and the perpendicular distance P of O from the tangent at Pare the $p - r$ coordinates of P. From Differential Geometry, we know that

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

Differentiating this with respect to θ,

$$-\frac{2}{p^3}\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} = 2\frac{du}{d\theta}\left[u + \frac{d^2u}{d\theta^2}\right].$$

If the central force (per unit mass) in the radial direction is $\varphi(r)$, then

$$\frac{d^2u}{d\theta^2} + u = -\frac{\varphi(r)}{h^2u^2}.$$

$$\therefore -\frac{2}{p^3}\frac{dp}{d\theta} = 2\frac{du}{d\theta}\left[-\frac{\varphi(r)}{h^2u^2}\right] \quad or \quad \frac{1}{p^3}\frac{dp}{d\theta} = \frac{\varphi(r)}{h^2u^2}\frac{du}{d\theta}.$$

$$\therefore \frac{1}{p^3}\frac{dp}{dr}.\frac{dr}{d\theta} = \frac{\varphi(r)}{h^2u^2}\frac{d\left(\frac{1}{r}\right)}{d\theta} \qquad Since\ u = \frac{1}{r}.$$

$$= \frac{\varphi(r)}{h^2u^2}\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right) = -\frac{\varphi(r)}{h^2}\frac{dr}{d\theta}.$$

$$\therefore \frac{h^2}{p^3}\frac{dp}{dr} = -\varphi(r).$$

This equation is in terms of P and r.This is the p − r equation of the orbit.This is also called the pedal equation of the orbit.

Equation for an attractive central force: If the central force is an attractive one of magnitude F per unit mass, then$\varphi(r) = -F$and the p − r equation of the orbit is

$$\frac{h^2}{p^3}\frac{dp}{dr} = F.$$

5.2.2 Law of a central force:

When the equation of a central orbit is given, to obtain the force per unit mass and the speed of the particle at a distance r from the centre of force, we have to calculate

$$-h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)\hat{r}\,,h\sqrt{\left(\frac{du}{d\theta}\right)^2 + u^2}.$$

5.2.3 Method to find the central orbit

If F is the central acceleration towards the centre of force, then to obtain the equation of the respective orbit, we have to solve the differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}. \qquad \ldots \ldots (1)$$

Here h may be found by using $h = pv$ from the given condition.

One method of solving the differential equation is by multiplying both sides of it by $2\frac{du}{d\theta}$ and then integrating with respect to θ. Then we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = 2\int \frac{F}{h^2u^2}\frac{du}{d\theta}d\theta$$

$$= 2\int \frac{F}{h^2u^2}du$$

because

$$\frac{d}{d\theta}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\left(\frac{d^2u}{d\theta^2} + u\right)\frac{du}{d\theta}.$$

Thus we get a first order differential equation

$$\frac{du}{d\theta} = (A function of u)$$

which can be solved as such or after replacing u by $\frac{1}{r}$.

Bookwork 5.4: To find the orbit of a particle moving under an attractive central force varying inversely as the square of the distance.

Let the pole be the centre of force. Let the force per unit mass be $\frac{\mu}{r^2}$. Then the differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2u^2}\frac{\mu}{r^2} or \frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}. \qquad \ldots \ldots (1)$$

Using the operator $\frac{d}{d\theta} = D, we\ have$

$$(D^2+1)u=\frac{\mu}{h^2}.$$

$$C.F.=A\cos(\theta+B); \quad P.I.=\frac{1}{D^2+1}\left(\frac{\mu}{h^2}\right)=\frac{\mu}{h^2}.$$

So the general solution is

$$u=A\cos(\theta+B)+\frac{\mu}{h^2} \qquad \ldots\ldots\ldots(2)$$

$$i.e., \quad \frac{h^2u}{\mu}=1+\frac{h^2A}{\mu}\cos(\theta+B)$$

$$i.e., \quad \frac{\left(h^2/\mu\right)}{r}=1+\frac{h^2A}{\mu}\cos(\theta+B).$$

It represents a conic whose semi latus rectum and eccentricity are

$$l=\frac{h^2}{\mu}, e=\frac{h^2A}{\mu}. \qquad \ldots\ldots(3)$$

Nature of the orbit: Multiplying both sides of (1) by $2\frac{du}{d\theta}$ and integrating with respect to θ, we get

$$\left(\frac{du}{d\theta}\right)^2+u^2=2\int\frac{\mu}{h^2}du \text{ or } \left(\frac{du}{d\theta}\right)^2+u^2=2\frac{\mu u}{h^2}+C$$

$$h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=2\mu u+c. \qquad \ldots..(4)$$

In this, the L.H.S. is the square of the velocity. Therefore

$$v^2=2\mu u+C. \qquad \ldots\ldots(5)$$

Substituting in (4) the value of u obtained from (2),

$$h^2\left[A^2\sin^2(\theta+B)+\frac{\mu^2}{h^4}+2.\frac{\mu}{h^2}.A\cos(\theta+B)+A^2\cos^2(\theta+B)\right]=2\mu u+C$$

Or $h^2\left[A^2+\frac{\mu^2}{h^4}+\frac{2\mu A}{h^2}\cos(\theta+B)\right]=2\mu u+C$

Or$h^2A^2 + \frac{\mu^2}{h^2} + 2\mu\left[u - \frac{\mu}{h^2}\right] = 2\mu u + C \qquad by(2)$

Or$h^2A^2 - \frac{\mu^2}{h^2} = C \;\; or \; h^2A^2 = \frac{\mu^2}{h^2} + C.$

$$\therefore \frac{h^4A^2}{\mu^2} = 1 + \frac{Ch^2}{\mu^2} \quad or \quad e^2 = 1 + \frac{Ch^2}{\mu^2} \quad by\ (3).$$

Thus the orbit is an ellipse or parabola or hyperbola according as

Ellipse	Parabola	Hyperbola
$e < 1$	$e = 1$	$e > 1$
$C < 0$	$C = 0$	$C > 0$
$v^2 - 2\mu u < 0$	$v^2 - 2\mu u = 0$	$v^2 - 2\mu u > 0$
$v^2 - \frac{2\mu}{r} < 0$	$v^2 = \frac{2\mu}{r} = 0$	$v^2 - \frac{2\mu}{r} > 0$
$v^2 < \frac{2\mu}{r}$	$v^2 = \frac{2\mu}{r}$	$v^2 > \frac{2\mu}{r}$
$v < \sqrt{\frac{2\mu}{r}}$	$v = \sqrt{\frac{2\mu}{r}}$	$v > \sqrt{\frac{2\mu}{r}}$

Therefore, the nature of the orbit depends on the velocity with which the particle is projected from the point whose distance from the centre of force is r.

Critical velocity: The quantity$\sqrt{\frac{2\mu}{r}}$is called the critical velocity at the distance r.So the nature of the orbit depends on the critical velocity.

The critical velocity can be seen to be the velocity that would be acquired by a particle, in the same force field, in reaching the point in question, starting from rest at the point at infinity.The equation for the motion from ∞ towards the pole is

$$\ddot{r} = -\frac{\mu}{r^2}$$

Multiplying both sides by $2\dot{r}$and then integrating w.r.t. t, we get

$$2\int \dot{r}\,\ddot{r}dt = -2\mu\int \frac{1}{r^2}dr \quad or \quad \dot{r}^2 = 2\mu\frac{1}{r} + C$$

Alternative method: The orbit and its nature can also be obtained from the differential equation

$$\frac{h^2}{p^3}\frac{dp}{dr} = F$$

which is in $p - r$coordinates.Now it is

$$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2} or h^2 \int \frac{dp}{p^3} = \mu \int \frac{dr}{r^2}$$

$$\frac{h^2}{-2p^2} = \frac{\mu}{-r} + A or \frac{h^2}{p^2} = \frac{2\mu}{r} + D \qquad \ldots..(6)$$

But we know that the p–r equations ofanellipse,aparabola,the nearer branch of a hyperbola are respectively

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1, \quad p^2 = ar, \quad \frac{b^2}{p^2} = \frac{2a}{r} + 1.$$

so, the orbit is an ellipse, a parabola or a hyperbola according as

$$D < 0, \quad D = 0, \quad D > 0. \qquad \ldots\ldots(7)$$

Also,we know that$pv = h$. Hence (6) becomes

$$v^2 = \frac{2\mu}{r} + D or D = v^2 - \frac{2\mu}{r}.$$

Thus the conditions (7) become that the orbit is an ellipse, a parabola, a hyperbola if

$$v < \sqrt{\frac{2\mu}{r}}, \quad v = \sqrt{\frac{2\mu}{r}}, \quad v > \sqrt{\frac{2\mu}{r}}.$$

Bookwork 5.5: To find the orbit of a particle moving under an attractive force varying as the distance.

Let the P.V. of the particle of mass m, at time t, be $\bar{r}$.If $n^2 r$ is the force per unit mass, then the equation of motion is

$$m\ddot{\bar{r}} = -mn^2 r\hat{r} \qquad or \quad \ddot{\bar{r}} = -n^2\bar{r}\,.$$

If $\bar{r} = x\bar{\imath} + y\bar{\jmath}$, then we get

$$\ddot{x} = -n^2 x, \qquad \ddot{y} = -n^2 y.$$

The general solutions of these differential equations are

$$x = Acos\,nt + Bsin\,nt,$$

$$y = C\cos nt + D\sin nt,$$

where the constants A,B,C,D depend upon the initial conditions.Solving these two equations for $\cos nt$,$sint\ nt$,

$$\frac{\cos nt}{\begin{vmatrix} B & -x \\ D & -y \end{vmatrix}} = \frac{-\sin nt}{\begin{vmatrix} A & x \\ C & y \end{vmatrix}} = \frac{1}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}}$$

$$\therefore \cos nt = \frac{-By + Dx}{AD - BC}, \sin nt = \frac{-Ay + Cx}{AD - BC}.$$

Squaring and adding, we get the equation of the path as

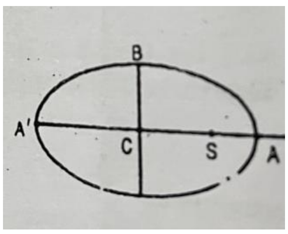
$$(Cx - Ay)^2 + (Dx - By)^2 = (AD - BC)^2$$

which, being a second-degree equation, represents a conic. Further it satisfies "$h^2 - ab < 0$" . Hence it represents an ellipse. Since$(-x, -y)$ satisfies the equation, the ellipse is symmetrical about the origin. So the centre of force is at the centre of the ellipse.

EXAMPLES

Example 1 : Show that the force towards the pole under which a particle describes the curve$r^n = a^n \cos n\theta$varies inversely as the $(2n + 3)^{th}$ power of the distance of the particle from the pole.

Solution:



Expressing the equation in terms of u,

$$\frac{1}{u^2} = a^n \cos n\theta.$$

Taking logarithm and differentiating w.r.t. θ,

$$-\frac{n}{u}\frac{du}{d\theta} = -\frac{1}{\cos n\theta}(n \sin n\theta) \quad or \quad \frac{du}{d\theta} = utan\, n\theta$$

$$\therefore \quad \frac{d^2u}{d\theta} = \frac{du}{d\theta}\tan n\theta + u(n \sec^2 n\theta)$$

$$= u \tan^2 n\theta + un \sec^2 n\theta = u(\tan^2 n\theta + \ n \sec^2 n\theta)$$

$$\therefore \quad h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = h^2u^3(\tan^2 n\theta + n \sec^2 n\theta + 1)$$

$$= h^2u^3(n+1)\sec^2 n\theta = h^2.\frac{1}{r^3}.(n+1)\frac{a^{2n}}{r^{2n}}$$

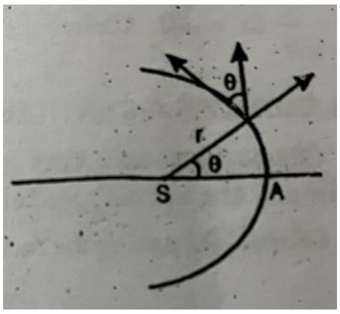
$$= \frac{h^2(n+1)a^{2n}}{r^{2n+3}},$$

This is the central attractive force which varies inversely asr^{2n+3}.

Example 2 : A particle moves with a central acceleration μ $r^{-7}$and starts from an apse at a distance a with a velocity equal to the velocity which would be acquired by the particle travellingfrom rest at infinity to the apse. Show that the equation of its orbit is

$$r^2 = a^2 cos2\theta.$$

Solution:



For the motion along the radius vector, from ∞ to any point P (r, θ)on the orbit, we have

$$\ddot{r} = -\frac{\mu}{r^7}. \qquad \dots\dots(1)$$

Let v be the velocity attained by the particle, travelling from rest at ∞ to P.Then multiplying both sides of (1) by 2 $\dot{r}$and integrating w.r.t. t,

$$[\dot{r}^2]_0^v = \int_\infty^r \frac{\mu}{r^7}\left(2\frac{dr}{dt}\right)dt = -2\mu\int_\infty^r \frac{1}{r^7}dr$$

$$v^2 = \frac{\mu}{3r^6}.$$

So the square of velocity v^2 of projection at the apse r = a is

$$v^2 = \frac{\mu}{3a^6}.$$

Next we shall find the value of the constant h^2.Initially at the apse r = p = a and from h = p v,

$$h^2 = p^2v^2 = a^2\left(\frac{\mu}{3a^6}\right) = \frac{\mu}{3a^4}. \qquad \ldots\ldots(2)$$

Now the differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2u^2r^7}.$$

Eliminating h^2 by (2) and simplifying,

$$\frac{d^2u}{d\theta^2} + u = 3a^4u^5.$$

Multiplying both sides by 2 $du/d\theta$ and integrating w.r.t. θ, we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \int 2(3a^4u^5)du$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = a^4u^6 + C.$$

But initially $du/d\theta = 0, u = 1/a$. Therefore $C = 0$ and

$$\left(\frac{du}{d\theta}\right)^2 = a^4u^6 - u^2$$

Since $u = 1/r, du/d\theta = -(1/r^2)\, dr/d\theta$.Therefore

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4}{r^6} - \frac{1}{r^3} or \frac{dr}{d\theta} = \frac{\sqrt{a^4 - r^4}}{r}$$

$$\frac{r}{\sqrt{a^4 - r^4}} dr = d\theta$$

Setting $r^2 = y$ and integrating,

$$\frac{1}{2}\frac{1}{\sqrt{a^4 - y^2}} dy = d\theta\,, \quad \sin^{-1}\frac{y}{a^2} = 2\theta + C.$$

$$\sin^{-1}\frac{r^2}{a^2} = 2\theta + C.$$

If the initial line is chosen through the apse so that the apse is $(a, 0)$,then when $r = a, \theta = 0$. This gives $C = \pi/2$.

$$\frac{r^2}{a^2} = \sin(2\theta + \frac{\pi}{2}) or r^2 = a^2 \cos 2\theta.$$

5.3 CONIC AS A CENTRAL ORBIT

Book work 5.6. When a central orbit is a conic with the centre of the force at one focus, to find the law of force and the speed of the particle.

Force: Choosing the focus S as the pole we get the polar equation of the conic as

$$\frac{l}{r} = 1 + e cos\theta or u = \frac{1}{l} + \frac{e}{l}\cos\theta$$

where l is the semilatus rectum. Therefore,

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\theta, \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

$$\therefore \; h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = h^2u^2\left[-\frac{e}{l}\cos\theta + \frac{1}{l} + \frac{e}{l}\cos\theta\right]$$

$$= h^2u^2\frac{1}{l} = \frac{h^2}{l}.\frac{1}{r^2}$$

Thus the force per unit mass in the radial direction but towards the pole is

$$\frac{h^2}{l}.\frac{1}{r^2}. \qquad \ldots..(1)$$

which is inversely proportional to the square of the distance from the pole.It is an attractive central force.

Inverse square law: In the above bookwork the force (1) is

$$\frac{h^2}{l}.\frac{1}{r^2}.$$

which is inversely proportional to the square of the distance.This rule of force is called inverse square law.

Velocity: The square of the speed of the particle at a distance r is

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$$

$$= h^2\left[\left(-\frac{e}{l}\sin\theta\right)^2 + \left(\frac{1}{l}+\frac{e}{l}\cos\theta\right)^2\right]$$

$$= \frac{h^2}{l^2}(e^2\sin^2\theta + 1 + 2e\cos\theta + e^2\cos^2\theta)$$

$$= \frac{h^2}{l^2}(e^2 + 1 + 2ecos\theta)$$

$$= \frac{h^2}{l^2}\left(e^2 + 1 + 2\left(\frac{1}{r} - 1\right)\right)$$

$$= \frac{h^2}{l^2}\left(e^2 + 1 + 2\frac{l}{r} - 2\right)$$

$$= \frac{h^2}{l^2}\left(e^2 - 1 + \frac{2l}{r}\right)$$

$$= \frac{h^2}{l}\left(\frac{e^2-1}{l} + \frac{2}{r}\right)$$

$$= \mu\left[\frac{2}{r} + \frac{e^2-1}{l}\right]$$

Where $\mu = \frac{h^2}{l}$

Parabola: When the path is a parabola $e = l$ And therefore

$$v^2 = \mu \frac{2}{r}$$

Ellipse: When the path is an ellipse,

$$v^2 = \mu \left[\frac{2}{r} + \frac{e^2 - 1}{l}\right]$$

$$= \mu \left[\frac{2}{r} + \frac{e^2 - 1}{b^2/a}\right]$$

$$= \mu \left[\frac{2}{r} + \frac{a(e^2 - 1)}{b^2}\right]$$

$$= \mu \left[\frac{2}{r} + \frac{a(e^2 - 1)}{ab^2}\right]$$

$$= \mu \left[\frac{2}{r} - \frac{b^2}{ab^2}\right] \qquad since\ b^2 = a^2(1 - e^2)$$

$$= \mu \left[\frac{2}{r} - \frac{1}{a}\right]$$

Maximum and minimum velocity: IfA, A' are the ends of the major axis that A is closer to S (the pole) than A', r is a minimum at A and maximum at A'. But the velocity is a maximum when r is a minimum. Thus the velocity is a maximum at A and a minimum at A'.

Periodic time: When the orbit is an ellipse the periodic time of the particle is the total area divided by the constant areal velocity. So it is

$$\frac{\pi ab}{h/2} = \frac{\pi ab}{\frac{1}{2}\sqrt{\mu l}} = \frac{2\pi ab\sqrt{a}}{\sqrt{\mu}b} = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}.$$

Hyperbola: When the path is the branch of the hyperbola nearer to the centre of force,

$$b^2 = a^2(e^2 - 1) or e^2 - 1 = \frac{b^2}{a^2}$$

and consequently

$$v^2 = \mu \left(\frac{2}{r} + \frac{1}{a}\right)$$

Remark: It is important to note that when the path is the branch of the hyperbola not nearer to the centre of force, its equation is

$$\frac{l}{r} = -1 + ecos\theta$$

In this case the force per unit mass is

$$\frac{h^2}{l}\frac{1}{r^2}r^{\wedge}$$

It is a repulsive force as we have in the case of like charges.

5.3.1 Kepler's laws of planetary motion

Before the development of mathematical theory of planetary motion by Newton, with the assumption that any two particles attract each other with a force$\gamma m_1 m_2/r^2$, where $\gamma$is a universal constant, m_1 and $m_2$are the masses of the particles and r is the distance between them, Kepler propounded the following three laws on the basis of astronomical observations:

K.1: The planets describe ellipses about the sun as focus.

K.2: The radius vector drawn from the sun to a planet sweeps out equal areas in equal times.

K.3: The squares of the periodic times of the planets are proportional to the cubes of the semimajor axes of their respective orbits.

It is obvious that K.2 implies that for each planet the areal velocity$\frac{1}{2}r^2\dot{\theta}$ is a constant and the transverse component of the force acting on the planet is zero. So the force acting on the planet is along the radius vector and thus it is a central force, the sun being the centre of force.K.1 implies that the force of attraction on the planet is inversely proportional to the square of its distance from the sun.So Newton's work is the assumption of inverse square law establishing the truth of Kepler's statement mathematically.

EXAMPLES

Example 1 : A particle describes an elliptic orbit under a central force towards one focus S.Ifv_1 is the speed at the end B of the minor axis and v_2, v_3 the speeds at the ends A, A' of the major axis, show that$v_1{}^2 = v_2 v_3$.

Solution:

In the present case the velocity formula is

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

Now the distances r of the points$A. A', B$ from S are:

$$SA = CA - CS = a - ae = a(1-e)$$

$$SA' = SC + CA' = ae + a = a(1+e)$$

$$SB = \sqrt{(ae)^2 + b^2} = a$$

$$\therefore \quad {v_1}^2 = \mu\left(\frac{2}{a} - \frac{1}{a}\right) = \frac{\mu}{a}$$

$${v_2}^2 = \mu\left\{\frac{2}{a(1-e)} - \frac{1}{a}\right\} = \frac{\mu}{a}\frac{2-1+e}{1-e} = \frac{\mu}{a}\frac{1+e}{1-e}$$

$${v_3}^2 = \mu\left\{\frac{2}{a(1+e)} - \frac{1}{a}\right\} = \frac{\mu}{a}\frac{2-1-e}{1+e} = \frac{\mu}{a}\frac{1-e}{1+e}$$

$${v_2}^2{v_3}^2 = \frac{\mu^2}{a^2} = {v_1}^4 or v_2 v_3 = v_1^2.$$

Example 2 : Show that the velocity of a particle moving in an ellipse about the centre of force at afocus is compounded of two constant velocities, namely,

i. $\frac{\mu}{h}$ perpendicular to the radius vector,
ii. $\frac{e\mu}{h}$ perpendicular to the major axis.

Solution:

If the equation of the ellipse is

$$\frac{l}{r} = 1 + ecos\theta, \qquad \ldots \ldots .. (1)$$

then we know that

$$\mu = \frac{h^2}{l} \quad or \quad l = \frac{h^2}{\mu} \qquad \ldots \ldots .. (2)$$

Also we have

$$r^2\dot{\theta} = h$$

The velocity components in the radial and transverse directions are

$$\dot{r}, r\dot{\theta}$$

Differentiating equation (1) w.r.t.t,

$$-\frac{1}{r^2}\dot{r} = -esin\theta\dot{\theta}$$

Using (3),

$$\dot{r} = \frac{esin\theta}{l}\left(r^2\dot{\theta}\right) = \frac{eh}{l}sin\theta by(3)$$

$$= \frac{e\mu}{h}sin\theta \quad by(2)$$

$$r\theta^{\cdot} = \frac{r^2\dot{\theta}}{r} = \frac{h}{r} \quad by(3)$$

$$= h\frac{1 + ecos\theta}{l} = \frac{\mu}{a}(1 + ecos\theta) \qquad by(2)$$

$$= \frac{\mu}{h} + \frac{e\mu}{h}\cos\theta \qquad \ldots\ldots(5)$$

From (4) and (5), we can say that the velocity of the particle is composed of the followingvelocities, namely,

$\frac{e\mu}{h}sin\theta$ in the radial direction, ……. (6)

$\frac{e\mu}{h}cos\theta$ in the transverse direction, ……. (7)

$\frac{\mu}{h}$ in the transverse direction.

From the figure it is clear that the radial and transverse directions are inclined to the perpendicular to the major axis at an angle θ and $90° - \theta$. So, the resultant of (6) and (7) is the velocity.

$\frac{e\mu}{h}$ perpendicular to the major axis …….. (9)

Now, (8) and (9) constitute the required result.

EXERCISES

1. The position vector of a particle at time t is $\bar{r} = \bar{a}cosnt + \bar{b}sinnt$ where $\bar{a}$ and $\bar{b}$ are constant vectors of n, a constant. Show that the particle is moving under a central attractive force varying as the distance.
2. A particle moves along the path $r = e^{\theta}$ under a central force. Show that the force is $\frac{2mh^2}{r^3}$ and speed of the particle is $\frac{h}{r}\sqrt{2}$.
3. In an orbit described under a force to a centre, the velocity at any point is inversely proportional to the distance of the point from the centre of the force. Show that the path is an equiangular spiral.
4. A particle describes a circular orbit under an attractive central force directed towards a point on the circle. Show that the force varies as the inverse fifth power of the distance.

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